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A Study on Analytic Functions Associated with the Generalized Hurwitz-Lerch zeta Function

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A B S T R A C T
In this study, we introduce and investigate new subclasses of analytic functions that are closely related to the generalized Hurwitz-Lerch zeta function. We establish certain inclusion relationships among these classes, and also consider the application of an associated integral operator within this context.

Keywords: *Analytic functions, linear operator, Hurwitz-Lerch zeta function.*

دراسة عن الدوال التحليلية المرتبطة بالتعميم الدالة زيتا هورويتز-ليرتش

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الملخص
يتناول هذا البحث تعريف ودراسة فئات فرعية جديدة من الدوال التحليلية ذات ارتباط وثيق بالدالة المعممة زيتا هورفيتز–ليرش. وقد تم التوصل إلى علاقات تضمين معينة بين هذه الفئات، كما تم النظر في تطبيق المؤثر التكاملي المرتبط بهذا السياق.

الكلمات المفتاحية: *الدوال التحليلية، المؤثر الخطي، الدالة زيتا هورويتز-ليرتش*

1. Introduction and Definitions

The geometric function theory in one complex variable represents a significant branch of complex analysis, with a rich history of foundational results. One of the landmark problems in this field was the Bieberbach conjecture, proposed in [1], which remained unsolved for decades until it was finally proven by the French mathematician de Branges in [2]. Although the conjecture itself has been resolved, the field continues to present numerous open problems that stimulate further investigation.

Linear operators, central to many areas of mathematics, also play vital roles in applied fields such as physics, control theory, dynamical systems, and various engineering disciplines. The theory of operators finds broad applications, particularly in solving differential and integral equations. In recent years, growing attention has been devoted to exploring differential and integral operators due to their significance across multiple research domains. Notably, linear operators serve as essential tools in the development of analytic function theory.

Numerous researchers have examined subclasses of analytic functions defined in the open unit disc through the application of linear operators. Building on this foundation, the present study aims to define, extend, and investigate certain linear operators acting on subclasses of analytic functions.

Let A represent the family of analytic functions given by the expression

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the unit disk
$U = \{z \in \mathbb{C}: |z| < 1\}$.

An integral operator was introduced and studied in 1999 within the context of analytic function theory, contributing to the development of new subclasses and their geometric properties.

$I_\rho: A \to A$ as follows:

$f_\rho(z) = \frac{z}{(1-z)^{\rho+1}}, (\rho \in N_0)$ and be $f_\rho(z)$ defined such that $f_\rho(z) * f_\rho^{(-1)}(z) = \frac{z}{(1-z)^2}$.

Then

$$I_\rho f(z) = \left[\frac{z}{(1-z)^{\rho+1}}\right]^{(-1)} * f(z).$$

We note that $I_0 f(z) = z f(z), I_1 f(z) = f(z)$. The operator I_ρ is called the Noor integral operator, which is an important tool in defining several classes of analytic functions. In recent developments, the Noor integral operator has demonstrated significant relevance within the framework of geometric function theory [3]. To facilitate the introduction of a broader class of linear operators, we employ a generalized form of the Hurwitz–Lerch zeta function, originally presented by Lin and Srivastava [4] and defined as follows:

$$F_{\delta,\mu}(a,s;z)=\psi_{\delta,\mu}^{(1,1)}(a,s;z)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{(\delta)_k(a+k)^s}z^k,$$

Where $\delta, a\neq 0,-1,-2,\dots, s\in\mathbb{C}$ when $|z|<1$ and $\Re(\delta+s-\mu)>0$, when $|z|=1$ and also $(v)_k$ denotes the Pochhammer symbol defined

$$(v)_k=\begin{cases}1, & k=0\\ v(v+1)\dots(v+k-1), & k\in N=\{1,2,3,\dots\}\end{cases}$$

Numerous researchers have explored various applications of the Hurwitz–Lerch zeta function across different mathematical contexts (see, for instance, [5], [6]-[9]).

We now define a function as follows:

$$G_{\delta,\mu}(a,s;z)=a^s zF_{\delta,\mu}(a,s;z)$$
$$=\sum_{k=1}^{\infty}\left(\frac{a}{a+k-1}\right)^s\frac{(\mu)_{k-1}}{(\delta)_{k-1}}z^k.$$

Also, we define a function $G_{\delta,\mu}^{(-1)}(a,s;z)$ as follows:

$$G_{\delta,\mu}(a,s;z)*G_{\delta,\mu}^{(-1)}(a,s;z)=\frac{z}{(1-z)^{\lambda}},\lambda>-1,$$

Here, we define a linear operator by the Hadamard product as follows:

$$L_{\lambda,\delta,\mu,q}(a,s,m)f(z)=G_{\delta,\mu}^{(-1)}(a,s;z)*(z+\sum_{k=2}^{\infty}([k]_q)^m a_k z^k)$$
$$=z+\sum_{k=2}^{\infty}([k]_q)^m\left(\frac{a+k-1}{a}\right)^s\frac{(\delta)_{k-1}(\lambda+1)_{k-1}}{(\mu)_{k-1}(1)_{k-1}}z^k, \quad (2)$$

where $[k]_q=\frac{1-q^k}{1-q}=1+q+q^2+\dots+q^{k-1}, 0<q<1$ and $m\in N_0$.Note that when $m=s=0$ and $\delta,\mu\in R$-{0,-1,-2,…} we obtain Cho-Kown-Srivastava operator $p=1$[10], when $\lambda=s=0$ we obtain Carlson Shaffer operator[11], when $m=s=\lambda=0,\delta=1$and $\mu=\rho+1,\rho\in N_0$ we obtain Noor integral operator introduced by Noor [12], when $s\in Z, a=2,\mu=\delta=1,\lambda=m=0$ we obtain The class of operators was investigated by Uralegaddi and Somanatha[13], In the particular case where s is a negative real number and the parameters take the values $a=2,\mu=\delta=1,m=0$the resulting operator coincides with the multipliertransformation previously studied by Jung et al.[14], when $\mu=\delta=1,\lambda=m=0,s=-\sigma\geq 0, a>0$, we obtain the integral operator [15], when $\mu=\delta=1,\lambda=m=0,s\in N_0, a>0$, we obtain analogously the multiplier transformation introduced by Cho et al. [16], when $a=\mu=\delta=1,\lambda=m=0$ and $s\in N_0$ we obtain the differential Salagean operator [17], when $\mu=\delta=1,s=m=0,\lambda\in N_0$ $\mu=\delta=1$, we obtain a differential operator defined by Ruscheweyh [18], and when $\mu=\delta=1$and $\lambda=m=0$ we obtain the Salagean type $q-$ differential operator [19].

By using the operator $L_{\lambda,\delta,\mu,q}(a,s,m)f(z)$

We now introduce the following classes of analytic functions

Definition 1 Let $f(z)\in A$, then $f(z)\in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ is a class of functions U if

$$\Re\left\{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)}\right\}>\gamma, \quad z\in U$$

where $0\leq\gamma<1$.

Definition 2 Let $f(z)\in A$, then $f(z)\in C_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ is a class of functions U if

$$\Re\left\{1+\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))''}{(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}\right\}>\gamma, \quad z\in U$$

where$0\leq\gamma<1$.

Remark: when $\mu=\delta=1,m=s=\lambda=0$, we have $S_{0,1,1,q}(a,0,0,\gamma)$ and $C_{0,1,1,q}(a,0,0,\gamma)$ was introduced by Robertson [20].

Definition 3 Let $f(z)\in A$ and $g(z)\in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ then $f(z)\in K_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$ is class of order $\beta$and type γ in$U$, $0\leq\gamma<1$ and $0\leq\beta<1$, if

$$\Re\left\{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)}\right\}>\beta, \quad z\in U.$$

Remark: when $\mu=\delta=1,m=s=\lambda=0$, we have $K_{0,1,1,q}(a,0,0,\gamma)$ was introduced by Libera [21].

Definition 4 Let $f(z)\in A$ and$g(z)\in C_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ then $f(z)\in K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$ is a class of functions of order $\beta$and type γ in$U$, $0\leq\gamma<1$ and $0\leq\beta<1$, if

$$\Re\left\{\frac{(z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))')'}{(L_{\lambda,\delta,\mu,q}(a,s,m)g(z))'}\right\}>\beta, \quad z\in U.$$

Remark: when $\mu=\delta=1,m=s=\lambda=0$, we have $K^*_{0,1,1,q}(a,0,0,\gamma,\beta)$ was introduced by Noor [22].

Now, we shall establish the inclusion relation. The following definition and lemmas are needed to establish our results.

Definition 5 [23] Let $f(z)\in A$ and $c\in N$. The generalized integral operator $F_c(f):A\to A$ is defined by

$$F_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(z)dt\ . \quad (3)$$

The special case $F_1(f)$ has been previously investigated in the works of Libera [24] and Livingston [25]

Lemma 6 [26] Let $\varphi(u,v)\in\mathbb{C}$ such that, $\varphi:D\to\mathbb{C}, D\subset\mathbb{C}\times\mathbb{C}$ and$u=u_1+iu_2$, $v=v_1+iv_2$

Suppose that $\varphi(u,v)$ satisfies the following:

1. $\varphi(u,v)$is continuous in D;
2. $(1,0)\in D$and $\Re\{\varphi(0,1)\}>1$;

3. $\Re\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in U, such that $(h(z), zh'(z)) \in D$ for all $z \in U$. If $\Re\{\varphi(h(z), zh'(z))\} > 0$, then $\Re\{h(z)\} > 0$ for $z \in U$.

Lemma 7[27] Let $\omega(z)$ be analytic in U with $\omega(0) = 0$, if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in U$, then we have $z_0\omega'(z_0) = k\omega(z_0)$, where $k \geq 1$.

2. Main results

The following results establish inclusion relationships for certain subclasses of analytic functions defined via the operator $L_{\lambda,\delta,\mu,q}(a,s,m)f(z)$ is as follows:

Theorem 1 $S_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma) \subset S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$

for $\Re a > 1-\gamma$ and $0 \leq \gamma < 1$.

Proof: Let $f(z) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ and set

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)} - \gamma = (1-\gamma)h(z)$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ Using the identity

$$z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))' = aL_{\lambda,\delta,\mu,q}(a,s+1,m)f(z) - (a-1)L_{\lambda,\delta,\mu,q}(a,s,m)f(z), \quad (4)$$

we have

$$\frac{L_{\lambda,\delta,\mu,q}(a,s+1,m)f(z)}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)} = \frac{1}{a}[(a-1)\gamma + (1-\gamma)h(z)], \quad (5)$$

By applying logarithmic differentiation to Eq. (5) with respect to z, we obtain:

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{(a-1)+\gamma+(1-\gamma)h(z)}$$

Taking $u = h(z), v = zh'(z)$ in (3) as

$$\varphi(u,v) = (1-\gamma)u + \frac{(1-\gamma)v}{(a-1)+\gamma+(1-\gamma)u} . \quad (6)$$

Consequently, from Eq. (6), it follows that:

1. $\varphi(u,v)$ is continuous in
$$D = \left(\mathbb{C} - \frac{a-1+\gamma}{\gamma-1}\right) \times \mathbb{C} ;$$
2. $(1,0) \in D$ and $\Re\{\varphi(0,1) > 1-\gamma > 0\ )\} > 1$;
3. for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$,

$$\Re\{\varphi(iu_2,v_1)\} = \frac{(\Re a - 1 + \gamma)(1-\gamma)(1+u_2^2)}{[(\Re a - 1)+\gamma)^2 + ((1-\gamma)^2 u_2 + \Im a)^2]}$$

$$\leq -\frac{1}{2}\frac{(\Re a - 1 + \gamma)(1-\gamma)v_1}{[(\Re a - 1)+\gamma)^2 + ((1-\gamma)^2 u_2 + \Im a)^2]} < 0.$$

for $\Re a > 1-\gamma, 0 \leq \gamma < 1$. Therefore, the function $\varphi(u,v)$ satisfies the conditions in Lemma 6. Then $f(z) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$.

Theorem 2

$$C_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma) \subset C_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$$

for $\Re a > 1-\gamma$ and $0 \leq \gamma < 1$.

Proof: $f(z) \in C_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma) \Leftrightarrow$

$L_{\lambda,\delta,\mu,q}(a,s+1,m)f(z) \in C_{0,1.1.q}(a,0,0,\gamma) \Leftrightarrow$

$L_{\lambda,\delta,\mu,q}(a,s+1,m)(zf'(z)) \in S_{0,1.1.q}(a,0,0,\gamma) \Leftrightarrow$

$zf'(z) \in S_{\lambda,\delta.\mu.q}(a,s+1,m,\gamma) \Leftrightarrow$

$zf'(z) \in S_{\lambda,\delta.\mu.q}(a,s,m,\gamma) \Leftrightarrow$

$L_{\lambda,\delta,\mu,q}(a,s,m)(zf'(z)) \in S_{0,1.1.q}(a,0,0,\gamma) \Leftrightarrow$

$z(L_{\lambda,\delta,\mu,q}(a,s,m)f'(z)) \in S_{0,1.1.q}(a,0,0,\gamma) \Leftrightarrow$

$L_{\lambda,\delta,\mu,q}(a,s,m)f(z) \in C_{0,1.1.q}(a,0,0,\gamma) \Leftrightarrow$

$L_{\lambda,\delta,\mu,q}(a,s,m)f(z) \in C_{\lambda,\delta.\mu.q}(a,s,m,\gamma)$

Inclusion Relations Involving Convex Analytic Functions

Theorem 3

$$K_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma,\beta) \subset K_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$$

for $\Re a > 1-\gamma, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$.

Proof: Let $f(z) \in K_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma)$ and $g(z) \in S_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma)$ such that $\Re\left\{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)}\right\} > \beta,\ z \in U.$

Now put

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} - \beta = (1-\beta)h(z) \quad (7)$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ By using Eq. (5), we have

$$= \frac{z(L_{\lambda,\delta,\mu,q}(a,s+1,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s+1,m)g(z)} \frac{L_{\lambda,\delta,\mu,q}(a,s+1,m)(zf'(z))}{L_{\lambda,\delta,\mu,q}(a,s+1,m)g(z)}.$$

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)zf'(z))' + (a-1)L_{\lambda,\delta,\mu,q}(a,s,m)(zf'(z))}{z(L_{\lambda,\delta,\mu,q}(a,s,m)g(z))' + (1-a)L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} \quad (8)$$

$$= \frac{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)zf'(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} + \frac{(a-1)L_{\lambda,\delta,\mu,q}(a,s,m)(zf'(z))}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)}}{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)g(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} + (1-a)}$$

Since $g(z) \in S_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma)$ and $S_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma) \subset S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ we let

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)g(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} = \gamma - (1-\gamma)H(z)$$

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)g(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} = \gamma + (1-\gamma)H(z),$$

where $\Re\{H(z)\} > 0$ and using Eq. (7). Thus, Eq. (8) can be written as

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s+1,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s+1,m)g(z)} =$$

$$= \frac{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)zf'(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} + (a-1)[\beta + (1-\beta)h(z)]}{\gamma + (1-\gamma))H(z) + (1-a)} \qquad (9)$$

Differentiating both sides of Eq. (7) and multiplying by z, we have

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)zf'(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} - \beta = (1-\beta)zh'(z) + \big(\beta + (1-\beta)h(z)\big).[\gamma + (1-\gamma))H(z)] \qquad (10)$$

Using Eq. (10) and Eq. (9), we have

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{\gamma + (1-\gamma))H(z) + (a-1)} \qquad (11)$$

Taking $u = h(z), v = zh'(z)$ in Eq. (11) as

$$\varphi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{(a-1) + \gamma + (1-\gamma)H(z)} \qquad (12)$$

It is evident that the function $\varphi(u,v)$ defined in $D \subset \mathbb{C} \times \mathbb{C}$ by Eq. (12) satisfies conditions (1) and (2). The verification of Lemma 6 up to this point is straightforward. To validate condition (3), we proceed as follows:

$$\Re\{\varphi(iu_2, v_1)\} = \frac{[(\Re a - 1) + \gamma + (1-\gamma)h_1(x,y)](1-\beta)v_1}{[(\Re a - 1) + \gamma + (1-\gamma)h_1(x,y)]^2 + (1-\gamma)(h_2(x,y) + \Im a)^2}$$

$$\leq \frac{-1}{2}\frac{[(c-1) + \gamma + (1-\gamma)h_1(x,y)](1-\beta)(1+u_2^2)}{[(c-1) + \gamma + (1-\gamma)h_1(x,y)]^2 + (1-\gamma)(h_2(x,y) + \Im a)^2} < 0,$$

for $\Re a > 1-\gamma$, where $H(z) = h_1(x,y) + ih_2(x,y)$ and $h_1(x,y), h_2(x,y)$ are functions of x, y, also $\Re H(z) = h_1(x,y) > 0$. Then $f(z) \in K_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$. Thus, the proof is finished.

Using a similar technique as in Theorem 3, and taking into account the fact that... $f(z) \in K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma) \Leftrightarrow zf'(z) \in K_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$, The following result is a direct consequence of Theorem 3.

Theorem4

$K^*_{\lambda,\delta,\mu,q}(a,s+1,m,\gamma,\beta) \subset K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$ for $\Re a > 1-\gamma\ 0 \leq \gamma < 1,$ and $0 \leq \beta < 1$.

Theorem 5 $S_{\lambda+1,\delta,\mu,q}(a,s,m,\gamma) \subset S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ for $\lambda > -1$ and $0 \leq \gamma < 1$.

Proof: Following a similar approach to that used in Theorem 2, and utilizing the identity:

$$z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))' = (1+\lambda)L_{\lambda+1,\delta,\mu,q}(a,s,m)f(z) - \lambda L_{\lambda,\delta,\mu,q}(a,s,m)f(z)$$

Thus, the proof is finished.

Theorem 6 $C_{\lambda+1,\delta,\mu,q}(a,s,m,\gamma) \subset C_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ for $\lambda > -1$ and $0 \leq \gamma < 1$.

Theorem 7

$K_{\lambda+1,\delta,\mu,q}(a,s,m,\gamma) \subset K_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ for $\lambda > -1, 0 \leq \beta < 1$and $0 \leq \gamma < 1$.

Theorem 8

$K^*_{\lambda+1,\delta,\mu,q}(a,s,m,\gamma) \subset K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma)\lambda > -1, 0 \leq \beta < 1$and $0 \leq \gamma < 1$.

Theorem 9

$S_{\lambda,\delta+1,\mu,q}(a,s,m,\gamma) \subset S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ for $\Re\delta > 1-\gamma$ and $0 \leq \gamma < 1$.

Proof: Following a similar approach to that used in Theorem 2, and employing the identity:

$$z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))' = \delta L_{\lambda,\delta+1,\mu,q}(a,s,m)f(z) - (\delta-1)L_{\lambda,\delta,\mu,q}(a,s,m)f(z)$$

Thus, the proof is finished.

Theorem 10

$C_{\lambda,\delta+1,\mu,q}(a,s,m,\gamma) \subset C_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$ for $\Re\delta > 1-\gamma$ and $0 \leq \gamma < 1$.

Theorem 11

$K_{\lambda,\delta+1,\mu,q}(a,s,m,\gamma,\beta) \subset K_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$for $\Re\delta > 1-\gamma, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$.

Theorem 12

$K^*_{\lambda,\delta+1,\mu,q}(a,s,m,\gamma,\beta) \subset K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma,\beta)$ for $\Re\delta > 1-\gamma, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$.

The remaining theorems can be proved using the same method, following the approach outlined above.

For $c > -1$and $f(z) \in A$. The following theorems are proved by using the integral operator $F_c(f)$ defined by Eq. (3).

Theorem 13 **Let** $c > -1$and $f(z) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$, then $F_c(f) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$.

Proof: From Eq. (12), we have

$$z(L_{\lambda,\delta,\mu,q}(a,s,m)F_c(f))' = (c+1)L_{\lambda,\delta,\mu,q}(a,s,m)f(z) - cL_{\lambda,\delta,\mu,q}(a,s,m)F_c(f), \qquad (13)$$

Set

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)F_c(f))'}{L_{\lambda,\delta,\mu,q}(a,s,m)F_c(f)} = \frac{1+(1-2\gamma)\omega(z)}{1-\omega(z)}, \qquad (14)$$

Where $\omega(z)$ is analytic in $U, \omega(0) = 0$. Using Eq. (13) and Eq. (14) we get

$$\frac{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)}{L_{\lambda,\delta,\mu,q}(a,s,m)F_c(f)} = \frac{C+1+(1-C-2\gamma)\omega(z)}{(C+1)(1-\omega(z))}. \qquad (15)$$

Differentiation Eq. (15) with respect to logarithmically, we obtain

$$\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z))'}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z)} = \frac{1+(1-2\gamma)\omega(z)}{1-\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + \frac{(1-c-2\gamma)z\omega'(z)}{c+1-(1-c-2\gamma)\omega(z)}. \qquad (16)$$

Now, we suppose $|\omega(z)| < 1, z \in U$ that.Conversely, there exists a point $z_0 \in U$ such that $\min_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)|$ .Then by lemma 7, we have $z_0\omega'(z_0) = k\omega(z_0)$ where $k \ge 1$. Putting $z = z_0$ and $\omega(z_0) = e^{i\theta}$ in Eq.(16), we have

$$\Re\left\{\frac{1+(1-2\gamma)\omega(z_0)}{1-\omega(z_0)}\right\} = \gamma \text{ , and}$$

$$\Re\left\{\frac{z(L_{\lambda,\delta,\mu,q}(a,s,m)f(z_0))'}{L_{\lambda,\delta,\mu,q}(a,s,m)f(z_0)}\right\} - \gamma$$
$$= \Re\left\{\frac{2(1-\gamma)ke^{i\theta}}{(1-e^{i\theta})(c+1-(1-c-2\gamma)e^{i\theta})}\right\}$$
$$= \Re\left\{\frac{-2k(1-\gamma)(c+\gamma)}{(1+c)^2+2(c+1)(1-c-2\gamma)cos\theta+(1-c-2\gamma)^2}\right\}$$
$$< 0,$$

which contradicts the hypothesis that $f(z) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$. Hence $|\omega(z)| < 1$ and it follows from Eq. (14) that $F_c(f) \in S_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$.

Theorem 14 Let $c > -1$. If $f(z) \in C(a,s,m,\gamma)$, then $F_c(f) \in C(a,s,m,\gamma)$.

Proof:

$$f \in C_{\lambda,\delta.\mu.q}(a,s,m,\gamma) \Leftrightarrow zf' \in S_{\lambda,\delta.\mu.q}(a,s,m,\gamma) \Leftrightarrow F_c(zf') \in S_{\lambda,\delta.\mu.q}(a,s,m,\gamma) \Leftrightarrow z(F_c(f))' \in S_{\lambda,\delta.\mu.q}(a,s,m,\gamma) \Leftrightarrow F_c(f) \in C_{\lambda,\delta.\mu.q}(a,s,m,\gamma).$$

Theorem 15 Let $c > -1$. If $f(z) \in K_{\lambda,\delta.\mu.q}(a,s,m,\gamma)$, then $F_c(f) \in K_{\lambda,\delta.\mu.q}(a,s,m,\gamma)$.

Proof: By employing a similar method to that used in the proof of Theorem 3, and utilizing both Theorem 13 and the identity given in Eq. (13). The proof of Theorem 15 has been completed.

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By adopting a similar approach, we obtain the following result:

Theorem 16 Suppose $c > -1$. If $f(z) \in K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$, then $F_c(f) \in K^*_{\lambda,\delta,\mu,q}(a,s,m,\gamma)$.

3. Conclusion

In this paper, A linear operator associated with certain subclasses of analytic functions involving the generalized Hurwitz–Lerch zeta function has been introduced. Several properties of the subclasses defined through this operator have been investigated. We conclude that this study, with some suggestions for future research, one direction is to study other classes of analytic functions involving our operator. Another direction would be constructed on other properties which are yet to be found, such as Fekete-Szegö ¨o problems, integral means for analytic functions with negative coefficients and convex linear combinations.

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