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Structural Properties of Graphs Defined by Bilinear Congruence on the Ring of Integers Modulo n

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A B S T R A C T

We investigate the structural properties of a graph defined on the ring $\mathbb{Z}_n \times \mathbb{Z}_n$. The Adjacency between two different vertices (a, b) and (x, y) is determined by the bilinear congruence $ay \equiv bx \pmod{n}$. We analyze three fundamental cases, $n = p^2, 2p$ and pq for distinct odd primes p, q. We describe the graph's breakdown into unit and non-unit vertex subsets. The unit subgraph forms disjoint cliques, with sizes depending on Euler's totient function. In contrast, the zero-divisor subgraph shows more complex behaviour governed by annihilation ideals. We establish general properties, including degree formulas, determination of maximum clique sizes in each component, determining the diameter, computing the girth, locating the graph centers, and finding the measures of vertex and edge connectivity. Additionally, we characterize independent sets and prove the existence of Hamiltonian cycles and supereulerian properties under certain connectivity conditions. Our results show how the prime factorization of n influences these properties.

Keywords: *Bilinear Form, Zero-Divisor Graph, Modular Arithmetic, Hamiltonian Cycle, Maximum Clique, Supereulerian Graph.*

الخصائص البنائية للرسوم البيانية المعرفة بواسطة التطابق الثنائي الخطي على حلقة الأعداد الصحيحة بمقياس n

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الملخص

ندرس الخصائص البنائية لرسم بياني مُعرَّف على الحلقة $\mathbb{Z}_n \times \mathbb{Z}_n$. يتم تحديد الإرتباط بين رأسين مختلفين (a, b) و (x, y) من خلال التطابق الثنائي الخطي $ay \equiv bx \pmod{n}$. نقوم بتحليل ثلاث حالات أساسية، وهي $n = p^2, pq, 2p$ لأعداد أولية فردية مختلفة p, q. نصف انقسام الرسم البياني إلى مجموعات رؤوس وحدة وغير وحدة. يشكل الرسم الجزئي للوحدات تجمعات (cliques) منفصلة ذات أحجام تعتمد على دالة أويلر. في المقابل، يُظهر الرسم الجزئي لقواسم الصفر سلوكا أكثر تعقيدا يخضع لمثاليات الفناء. نثبت خصائص عامة تشمل درجة الرؤوس، وتحديد أحجام أعظم تجمعات في كل مكون، تحديد قطر بيان الرسم، وحساب مقاس الرسم، وتحديد مراكز الرسم، وإيجاد مقاييس اتصال الرؤوس والحواف. بالإضافة إلى ذلك، نقوم بوصف المجموعات المستقلة ونثبت وجود دورات هاميلتونية وفائقة الأويلرية تحت شروط اتصال معينة. تُظهر نتائجنا كيف يؤثر التحليل الأولي للعدد n على هذه الخصائص .

الكلمات المفتاحية: صيغة ثنائية خطية، الرسم البياني لقواسم الصفر، الحساب النمطي، دورة هاميلتونية، أكبر تجمّع، رسم بياني فائق أويلري.

1. Introduction

The study of graphs based on algebraic structures has provided deep insight into the connection between algebra and graph theory. Among the most important types are zero-divisor graphs [1], where vertices represent elements of a commutative ring, and edges connect pairs whose product is zero. These graphs capture valuable information about ring properties, including ideal structure, annihilators, and zero-divisor behavior. A related model is the dot product graph [2], which is defined on $\mathbb{Z}_n^k$ with adjacency determined by orthogonal vectors under the dot product modulo n. Such graphs have applications in linear algebra over finite rings

and combinatorial design theory. A third type arises from symplectic graphs [3], which are formed from alternating bilinear forms over finite fields and show high symmetry, with applications in geometry and coding theory.

In this paper, we introduce a new graph, denoted as G_n, defined on $\mathbb{Z}_n \times \mathbb{Z}_n$, where two distinct vertices (a, b) and (x, y) are adjacent if and only if $ay \equiv bx \ (mod\ n)$. This adjacency condition generalizes the determinant criterion for linear dependence in $\mathbb{Z}_n \times \mathbb{Z}_n$ as $ay - bx \equiv 0 \ (mod\ n)$ implies (a, b) and (x, y) are linearly dependent. It also provides a modular version of skew-symmetric bilinear forms. The graph combines important features of three previous models. Like zero-divisor graphs, it shows annihilation properties. Like dot product graphs, it represents linear dependence. Like symplectic graphs, it comes from a bilinear form that has inherent orthogonality. Importantly, this graph offers a way to examine how modular arithmetic affects graph properties.

The algebraic distinction between units and zero divisors in $\mathbb{Z}_n$ naturally partitions G_n into two subgraphs: one formed by units of $\mathbb{Z}_n$ and another by zero divisors. The unit subgraph H_n exhibits symmetry and regularity, decomposing into disjoint cliques whose sizes depend on Euler's totient function $\phi(n)$. On the other hand, the zero-divisor subgraph Γ_n reveals more complex behavior, shaped by the interplay of ideals and annihilators in $\mathbb{Z}_n$. This difference helps us analyze the graph's topology through algebraic perspectives, showing how prime factorization affects connectivity, clique formation, and cycles.

For $n = p^2$, $n = 2p$, and $n = pq$, where p, q are distinct odd primes; we show that H_n decomposes to cliques sized by $\phi(n)$ and Γ_n exhibits annihilation-driven connectivity. For each case, we derive exact degree formulas, characterize connected components, and determine maximal cliques. We also compute global measures such as diameter, girth, and vertex/edge connectivity. A key contribution is the identification of Hamiltonian cycles and supereulerian properties under specific connectivity conditions, linking algebraic constraints (e.g., $\kappa(\Gamma_n) \geq \alpha(\Gamma_n)$ to combinatorial phenomena. Notably, for $n = p^2$, we show that Γ_n becomes a complete graph K_{p^2}, while for $n = pq$, the graph's complexity reflects the multiplicative structure of the Chinese Remainder Theorem.

To carefully examine these graph structures, we start by reviewing basic concepts in ring theory and graph theory that support our study. This math and counting tools will be crucial for grasping the adjacency conditions and connectivity patterns in G_n.

2. Background

In this section, we recall some basic concepts in ring theory [4-8], focusing on the ring of integers modulo n, denoted $\mathbb{Z}_n$. Recall that an element a in $\mathbb{Z}_n$ is a *unit* if $\gcd(a, n) = 1$, and *a zero divisor* if $\gcd(a, n) > 1$(excluding zero). The set of units forms a multiplicative group, denoted as $\mathbb{U}_n$, while zero divisors together with zero, denoted as $\mathbb{D}_n$, has interesting algebraic properties.

Closely related to zero divisors is the notion of an *annihilator*. For an element a in a commutative ring $\mathbb{Z}_n$, the annihilator of a, denoted $ann(a)$, is the set of all elements $r \in \mathbb{Z}_n$ such that $r \cdot a = 0$. This set forms an ideal of $\mathbb{Z}_n$, capturing algebraic obstructions to a's invertibility. When a is a zero divisor, $ann(a)$ is non-trivial (i.e., contains non-zero elements), while for units, $ann(a)$ collapses to $\{0\}$. In $\mathbb{Z}_n$, the structure of $ann(a)$ is explicitly determined by $gcd(a, n)$. This ties directly to the Euler phi function $\phi(n)$, which counts the number of units in $\mathbb{Z}_n$, equivalently, the order of the group of units $\mathbb{U}_n$. These algebraic properties directly influence our graph construction, where adjacency is determined by a bilinear form over $\mathbb{Z}_n \times \mathbb{Z}_n$.

Standard graph notation and terms follow [9-12], and any extra conventions will be clearly defined when they are introduced. A vertex v is *adjacent* to u if the edge $u\,v$ exists. The neighborhood of u is the subgraph induced by all vertices adjacent to u. A *connected* graph is a graph where a path exists between any two vertices. A *spanning subgraph* of G retains all vertices but may omit edges. An *acyclic* graph is a graph that contains no cycles, meaning no path starts and ends at the same vertex while traversing distinct edges. If such a subgraph is acyclic and connected, it is a spanning tree. *A vertex cut* is a vertex subset whose removal disconnects G, with the smallest such set defining the vertex connectivity $\kappa(G)$. Similarly, *an edge cut* disconnects G when removed, and the edge connectivity $\kappa'(G)$ is the minimal size of such a cut. The Minimum degree $\delta(G)$ is the smallest degree of any vertex in the graph. Thus, removing all edges incident to a minimum-degree vertex disconnects it, so $\kappa'(G)$ cannot exceed $\delta(G)$. *A clique* is a subset of mutually adjacent vertices represents a complete subgraph. The *clique number* $\omega(G)$ is the maximum number of vertices along the complete subgraphs of

G. The *eccentricity* of a vertex v in a connected graph G is the maximum graph distance between v and any other vertex u of G. For a disconnected graph, all vertices are defined to have infinite eccentricity. The *diameter* is the greatest distance (maximum eccentricity) between any two vertices, the *girth* is the shortest cycle length, and the graph *centre* is the vertex with minimal eccentricity. *Independent sets*, where no two vertices are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set, and its size is called the independence number of G and is usually denoted by $\alpha(G)$. *An Euler trail* traverses every edge exactly once, while a closed version is an Eulerian circuit.

The following sufficient conditions for Hamiltonian and supereulerian graphs will be critical for analyzing Γ_n in composite cases (see Propositions 3.8 and 3.12):

Theorem 2.1. [12]. A connected graph G is Eulerian if and only if all vertices have even degree.

In contrast, *a Hamilton path* visits every vertex exactly once, and *a Hamilton cycle* completes this traversal into a closed cycle.

One of the most influential sufficient conditions for Hamiltonian graphs was introduced by Chvátal and Erdős. The following sufficient conditions will be critical for analyzing Hamiltonicity and supereulerian properties in Section 3.

Theorem 2.2. [13]. Let G be an undirected graph. If $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.

The following theorem, due to Bang-Jensen, and Alessandro [14], provides a sufficient condition for an undirected graph to be supereulerian.

Theorem 2.3. Let G be an undirected graph on at least three vertices. If $\lambda(G) \geq \alpha(G)$, then G is supereulerian.

This resolves the undirected case of a broader conjecture. It proves that such graphs always contain a spanning closed trail. Catlin's foundational survey [15] established key properties and sufficient conditions for supereulerian graphs, along with reduction techniques. His conjecture that 3-edge-connected graphs with $\alpha(G) \leq 2$ are supereulerian. Catlin later proved this [16], which improved the criteria based on connectivity. Han et al. studied a weaker sufficient condition for supereulerian graphs. They proved that if $\kappa(G) \geq \alpha(G) - 1$, then G must be either supereulerian or part of a certain infinite family of exceptions.

The vertex set of our graph splits into two independent subsets based on the algebraic structure of $\mathbb{Z}_n$. Let $\mathbb{U}_n$ denote the group of units in $\mathbb{Z}_n$ and $\mathbb{D}_n$ the set of zero divisors (including zero). We define the subgraph H_n with the vertex set $\mathbb{U}_n \times \mathbb{U}_n$ which consists of pairs where both components are units. We also define the subgraph Γ_n with the vertex set $\mathbb{D}_n \times \mathbb{D}_n$ consisting of pairs where both components are zero divisors. This partition into unit and zero-divisor subgraphs will support the clique decomposition of H_n (Section 3.1) and the annihilation-driven structure of Γ_n (Section 3.2).

In H_n, the adjacency relation shows multiplicative properties of units, while Γ_n highlights annihilation relations between zero divisors. By looking at these subgraphs separately, we can see how the different algebraic properties of units and zero divisors influence their specific graph structures.

The graph in this work is created using pairs $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$, where adjacency is determined by the bilinear relation:

$$(a, b) \sim (x, y) \quad \text{if and only if } a\,y \equiv b\,x \pmod{n}.$$

This condition arises naturally from a bilinear form $\mathcal{B}: A \times A \to \mathbb{Z}_n$, where $A = \mathbb{Z}_n \times \mathbb{Z}_n$ defined as:

$$\mathcal{B}((a, b), (x, y)) = a\,y - b\,x.$$

Here, two vertices (a, b) and (x, y) are connected precisely when the bilinear form evaluates to zero modulo n.

The bilinear form $\mathcal{B}((a, b), (x, y)) = ay - bx$ is skew-symmetric $\mathcal{B}(v, w) = -B(w, v)$ and linear in each argument (see [17]). This induces symmetric adjacency in G_n. Thus, the graph is simple.

Unlike classical zero-divisor graphs [1] or dot-product graphs [2], it combines annihilation $ay \equiv bx \pmod{n}$ with bilinear dependence, yielding new symmetry properties explored in section 3.

3. Results

In this section, we organize the results into different cases based on the factorization of n. This includes $n = p^2, n = 2p$, and $n = pq$ for distinct odd primes p and q. We describe how the graph breaks down into unit and zero-divisor subgraphs. We also derive precise degree formulas and explore connectivity, clique structures, and independence number. Important findings include identifying maximum cliques, calculating diameter and girth, and analyzing conditions for Hamiltonicity and supereulerian properties.

1.1. The graph of units H_n

Case 1: If $n = p$ is an odd prime.

When n is an odd prime, the ring $\mathbb{Z}_p$ is a field, meaning every non-zero element is a unit. The graph H_n on $\mathbb{U}_p \times \mathbb{U}_p$ exhibits a highly symmetric structure due to the invertibility of its elements and

the bilinear relation applied. The following proposition describes its decomposition into disjoint cliques.

Since units in $\mathbb{Z}_p$ are closed under multiplication, adjacency in H_p reduces to scalar multiples, inducing a clique structure

Proposition 3.1. If p is an odd prime, the graph H_p decomposes into $(p-1)$ disjoint components of the complete graph K_{p-1}.

Proof. Since $\mathbb{Z}_p$ is a field, the adjacency condition $ay \equiv bx \ (mod\ p)$ holds if and only if (a,b) and (x,y) are scalar multiples, i.e., $(x,y) = k(a,b)$ for some $k \in \mathbb{U}_p$. These partitions $\mathbb{U}_p \times \mathbb{U}_p$ into equivalence classes where each class consists of all non-zero scalar multiples of a fixed pair (a,b). Each equivalence class forms a clique K_{p-1}, as any two distinct vertices in the same class are adjacent. Since there are $(p-1)$ distinct non-zero scaling factors in $\mathbb{Z}_p$, the graph decomposes into $(p-1)$ disjoint cliques. Thus, $\mathrm{H}_n \cong (p-1)K_{p-1}$. ■

Corollary 3.1. For $n = p$, the unit subgraph H_n has edge count:

$$|E(\mathrm{H}_n)| = (p-1)\sum_{i=1}^{p-1}(p-1) - i.$$

Since we have defined the unit subgraph for prime n, we now extend our discussion to composite $n = pq$. In this case, the Chinese Remainder Theorem adds more structure, resulting in larger cliques identified by Euler's totient function.

Case 2: If $n = p\,q$ is a composite.

When $n = p\,q$ for distinct odd primes p and q, the structure of H_n shows the properties of units in the ring $\mathbb{Z}_{pq}$. The following proposition describes its decomposition into larger disjoint cliques.

Proposition 3.2. If $n = p\,q$ for distinct odd primes p and q, the graph H_n decomposes into $(p-1)(q-1)$ disjoint cliques of the complete graph $K_{(p-1)(q-1)}$.

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Thus, the group of units $\mathbb{U}_n$ has order $\phi(n) = (p-1)(q-1)$ and every unit $k \in \mathbb{U}_n$ is uniquely determined by its residues modulo p and q. For any two vertices (a,b) and (x,y) in $\mathbb{U}_n \times \mathbb{U}_n$, the adjacency condition $ay \equiv bx \ (mod\ pq)$ is satisfied precisely when (x,y) is a scalar multiple of (a,b) by some unit $k \in \mathbb{U}_n$. This occurs because in $\mathbb{Z}_n$ the modular bilinear is equivalent to the linear dependence condition $x \equiv ka$ and $y \equiv kb$ for some unit k.

For any fixed vertex $v = (a,b) \in \mathbb{U}_n \times \mathbb{U}_n$, define its proportionality class as follows:

$$[v] = \{kv : k \in \mathbb{U}_n\}$$

Since all vertices in $[v]$ are scalar multiples, then, they form a complete subgraph $K\_(p-1)(q-1)$. Moreover, there are exactly $(p-1)(q-1)$ such disjoint cliques. ■

Corollary 3.2. For $n = pq$, the unit subgraph H_n has edge count:

$$|E(\mathrm{H}_n)| = (p-1)(q-1)\sum_{i=1}^{(p-1)(q-1)}(p-1)(q-1) - i.$$

Case 3: If $n = p^2$ for an odd prime p.

When $n = p^2$, the unit subgraph H_n has a special structure. Unlike the cases $n = p$ or $n = pq$, units in $\mathbb{Z}_{p^2}$ can be expressed in two parts: a unit modulo p plus a multiple of p. The adjacency condition $ay \equiv bx \ (mod\ p^2)$ creates cliques based on proportional pairs, but only within distinct directions modulo p. This leads to the following precise decomposition.

Proposition 3.3. Let $n = p^2$ where p is an odd prime. The graph H_n decomposes into $p(p-1)$ disjoint cliques, each isomorphic to K_{p-1}.

Proof: The group of units $\mathbb{U}_{p^2}$ has order $\phi(p^2) = p(p-1)$. By the Chinese Remainder Theorem, every unit $u \in \mathbb{U}_{p^2}$ can be expressed uniquely as:

$$u \equiv a + bp \ (mod\ p^2), \ \text{where } a \in \{1, \dots, p-1\}, b \in \{0, \dots, p-1\}.$$

Here, a is a unit modulo p, and b parametrizes the lift to $\mathbb{Z}_{p^2}$

Two vertices $(a,b),(x,y) \in \mathbb{U}_{p^2} \times \mathbb{U}_{p^2}$ are adjacent in H_n if and only if:

$$ay \equiv bx \ (mod\ p^2).$$

For units, this simplifies to the proportionality condition $(x,y) \equiv k(a,b) \ (mod\ p^2)$ for some $k \in \mathbb{U}_{p^2}$. Now, fix a vertex $v = (a,b)$. Its scalar multiple class is:

$$[v] = \{kv : k \in \mathbb{U}_{p^2}\}.$$

By considering residues modulo p, there are exactly $p-1$ distinct values of $k \ (mod\ p)$ because $a \not\equiv 0 \ (mod\ p)$. Each such k can be lifted to p possible values modulo p^2, but only $p-1$ are units in $\mathbb{Z}_{p^2}$. Consequently, the class $[v]$ forms a complete subgraphK_{p-1}, and there are $\phi(p^2) = p(p-1)$ such disjoint cliques, corresponding to each distinct proportionality class v modulo p.

Crucially, if two vertices (a,b) and (x,y) are not scalar multiples modulo p, then $ay \not\equiv bx \ (mod\ p)$, which implies $ay \not\equiv bx \ (mod\ p^2)$. Thus, no edges exist between distinct cliques. ■

Corollary 3.3. For $n = p^2$, the unit subgraph H_n has edge count:

$$|E(\mathrm{H}_n)| = p(p-1)\sum_{i=1}^{p(p-1)} p(p-1) - i.$$

Example 3.1. For $p = 3$, H_9 consists of 6 disjoint copies of K_6 (edges) as shown in Figure 1. Units in $\mathbb{U}_9$ are $\{1, 2, 4, 5, 7, 8\}$, and adjacency holds if and only if $(x, y) \equiv k(a, b)\ (mod\ 9)$ for $k \in \mathbb{U}_9$.

Remark 3.1. In all three cases mentioned, the unit subgraph H_n shows:

i. Disconnected components (cliques) mean that $diam(u, v) = \infty$ for vertices in different cliques.
ii. Each clique K_{p-1}, $K_{(p-1)(q-1)}$ and $K_{p(p-1)}$ includes triangles but has no cycles shorter than 3. i.e., $girth(\mathrm{H}_n) = 3$.

This uniformity occurs because adjacency in H_n is based only on scalar multiples within disjoint equivalence classes.

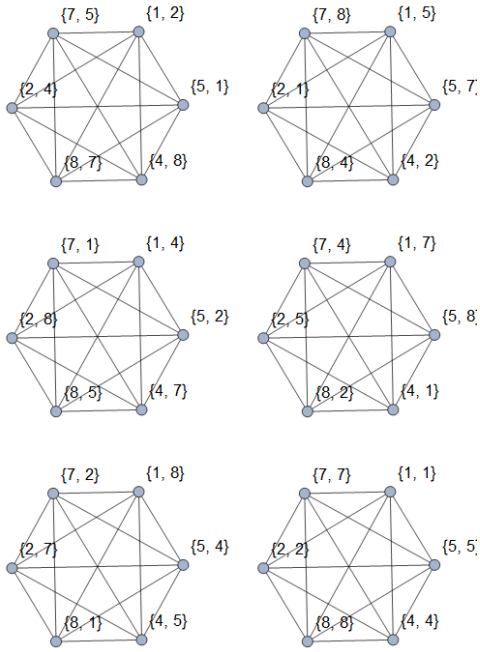

Figure 1: The graph of Units H_9.

While the unit subgraph H_n shows clear clique decompositions, the zero-divisor subgraph Γ_n has more complexity because of annihilation relations. We will now analyze Γ_n, where adjacency represents ideal interactions instead of multiplicative inverses.

1.2. The graph of zero divisors Γ_n

In this section, we examine the graph Γ_n, which includes pairs of zero divisors in the ring $\mathbb{Z}_n \times \mathbb{Z}_n$. Adjacency is defined by the bilinear form $\mathcal{B}$. This bilinear congruence captures modular orthogonality and shows how zero divisors interact. For composite n, the structure of Γ_n is strongly affected by the prime factorization of n. This leads to distinct connectivity patterns, clique formations, and maximal independent sets.

Case 1: If $n = 2p$, where p is an odd prime.

Let p is an odd prime. The set of zero divisors $\mathbb{D}_n$ can be written as $\mathbb{D}_n = M_2 \cup M_p$, where $M_2 = \{0, 2, 4, 6, \ldots, 2p-2\}$, is the principal ideal generated by 2 in $\mathbb{Z}_n$, and $M_p = \{0, p\}$. Furthermore, we refer to $M_2/\{0\}$ by M_2^* and the set $M_p/\{0\}$ by M_p^*. The sets M_2 and M_p form additive subgroups of $\mathbb{Z}_n$, and they exhibit an annihilator duality:

i. Every element $k \in M_2$ satisfies $k \cdot p \equiv 0\ (mod\ n)$, meaning $M_2 \subseteq ann(p)$.
ii. Conversely, p annihilates all elements of M_2, i.e., $M_p \subseteq ann(k)$ for any $k \in M_2$.

This structure highlights the interplay between the ideals M_2 and M_p, where each subgroup consists precisely of the annihilators of the other.

We partition the set of vertices $\mathbb{D}_{2p} \times \mathbb{D}_{2p}$ into the following subsets:

$$S_0 = \{(0,0), (0,p), (p,0), (p,p)\},$$
$$S_1 = \{(a_i, 0): a_i \in M_2^*, 1 \le i \le p-1\},$$
$$S_2 = \{(0, a_i): a_i \in M_2^*, 1 \le i \le p-1\},$$
$$S_3 = \{(a_i, a_i): a_i \in M_2^*, 1 \le i \le p-1\},$$
$$S_4 = \{(a_i, b_i): a_i, b_i \in M_2^*, 1 \le i \le p-1\}.$$
$$S_5 = \{(p, a_i): a_i \in M_2^*, 1 \le i \le p-1\},$$
$$S_6 = \{(a_i, p): a_i \in M_2^*, 1 \le i \le p-1\},$$

We now examine the zero-divisor graph Γ_{2p}, starting with a classification of vertex degrees. This basic analysis shows how annihilation relations in $\mathbb{Z}_n$ determine connectivity patterns in Γ_{2p}.

Proposition 3.4. For $a, b \in M_2^*$ the degree of a vertex $v \in V(\Gamma_{2p})$ are classified as follows:

$$\deg(v) = \begin{cases} p^2 + 2p & if\ v = (0,0) \\ 2p + 1 & if\ v = (a,0)\ or\ (0,a) \\ 2p - 1 & if\ v = (a,p)\ or\ (p,a) \\ p^2 + p - 1 & if\ v = (p,0)\ or\ (0,p) \\ p + 2 & if\ v = (a,b) \\ p^2 & if\ v = (p,p) \end{cases}$$

Proof: By definition, the vertex $v = (0,0)$ is adjacent to each single vertex in $V(\Gamma_n)$, and since the number of vertices in this graph is $|V(\Gamma_n)| = \big(n - \phi(n)\big)^2 = (p+1)^2 = p^2 + 2p + 1$. Since the adjacency relation is defined on different vertices, then we have

$$\deg(v) = p^2 + 2p.$$

For $a \in M_2^*$, let $v = (a, 0)$, then v is adjacent to all vertices with linear dependence, i.e., to each vertex in $S_1' = S_1/\{(a, 0)\}$. Also, v is adjacent to vertices with annihilator entries, i.e., S_0, where $0, p \in ann(a)$. Moreover, v is adjacent to the vertices in S_6, Since $|S_1'| = p - 2$, $|S_0| = 4$, and $|S_6| = p - 1$. Thus,

$$\deg(v) = (p - 2) + (p - 1) + 4 = 2p + 1.$$

In the same way, the degree of the vertex $v = (0, a)$ is $2p + 1$.

Consider $v = (a, p)$, then v is adjacent to vertices with linear dependence, i.e., to vertices in $S_6' = S_6/\{(a, p)\}$, and to the vertices with annihilator entries, i.e., $S_0' = S_0/\{(p, 0), (p, p)\}$. The third type of adjacent vertices to v is mixed vertices S_1, Since $|S_6'| = p - 2$, $|S_0'| = 2$, and $|S_1| = p - 1$. Thus,

$$\deg(v) = (p - 2) + (p - 1) + 2 = 2p - 1.$$

Likewise, the degree of the vertex $v = (p, a)$ is $2p - 1$.

Let $v = (p, 0)$, since $ann(p) = M_2$, then the vertex v is adjacent to vertices with annihilator entries, i.e., S_4, and to the mixed vertices in the set S_5. Since $|S_4| = p^2$, $|S_5| = p - 1$. Thus,

$$\deg(v) = p^2 + p - 1$$

Similarly, the degree of the vertex $v = (0, p)$ is $p^2 + p - 1$.

Now, we investigate the degree of the vertex $v = (p, p)$. The vertex v is adjacent to the vertices in the set S_1, S_2, S_3, S_4 and the vertex $(0, 0)$, That means there are $3(p - 1) + (p^2 - 3p + 2) + 1 = p^2$ vertices adjacent to this vertex. Since (p, p) is not adjacent to (a, p) nor to (p, a) for any $a \in M_2$, because the adjacency condition fails. Thus,

$$\deg(v) = p^2.$$

Finally, let $v = (a, b)$ for any $a, b \in M_2^*$. It is clear that (a, b) is adjacent to the vertices in the set S_0. By the definition of the graph Γ_{2p}, v is adjacent to vertices with multiple coordinates (ma, mb), which are $p - 2$. Thus,

$$\deg(v) = 4 + (p - 2 = p + 2. \blacksquare$$

Corollary 3.4. The edge connectivity of Γ_{2p}, is $\kappa'(\Gamma_n) = p + 2$.

Proposition 3.5. The graph Γ_{2p} satisfies the following properties:

i. The vertex (0,0) is the unique centre, with eccentricity 1.
ii. The diameter of Γ_n is $diam(\Gamma_n) = 2$.
iii. The girth of Γ_n is $girth(\Gamma_n) = 3$.

Proof: (i.) Consider $v = (0, 0)$. Since v is adjacent to all vertices in the graph Γ_{2p}, then the eccentricity of v is 1, and for any other vertex (a, b), its eccentricity is at least 2 from some nonadjacent vertex (x, y). Hence, v is the unique centre.

(ii.) Since $v = (0, 0)$ is a unique center of the graph Γ_n with eccentricity 1, then for any two nonadjacent vertices (a, b) and (x, y), they share v as a common neighbour. Thus, the greatest distance between them is 2, which proves (ii).

(iii.) From the definition of $\mathcal{B}$, the graph Γ_{2p} is simple with no multiple edges, so $girth(\Gamma_{2p}) > 2$. For any adjacent vertices (a, b) and (x, y), the set $[\{(0, 0), (a, b), (x, y)\}]$ induces a 3-cycle, which is the smallest possible. Hence, the proof follows. ■

Proposition 3.6. For any fixed $a \in M_2^*$, the set

$$I_a = \{(a, b): b \in M_2\} \cup \{(0, a)\}$$

is a maximum independent vertex set in Γ_{2p} with independence number $\alpha(\Gamma_{2p}) = p + 1$.

Proof: For any two vertices (x, y) and (a, b) in I_a, adjacency requires $a\,y \equiv b\,x \pmod{2p}$, which holds only if they are linearly independent. For (a, b) and $(0, a)$, adjacency requires $a \cdot a \equiv b.0 \equiv \pmod{2p}$, i.e., $0 \equiv a^2 \pmod{2p}$. Since $0 \not\equiv a^2 \pmod{2p}$ for any $a \in M_2^*$. Thus, (a, b) and $(0, a)$ are not adjacent. Since all vertices in $\{(a, b): b \in M_2\}$ are linearly independent, and no annihilators are involved in $\{(a, b): b \in M_2\}$. Thus, distinct vertices in this set are non-adjacent.

To prove that I_a is the maximum, consider adding any vertex $(x, y) \notin I_a$. Thus, $(x, y) \notin \{\{0,0), (0, p), (p, 0), (p, p), (a, p), (p, a)\}$, which implies $(x, y) \in \{(m\,a, a): a \in M_2^*\}$, for some positive integer m, then an edge will be created with at least one vertex in I_a such as $(m\,a, a)$ and $(a, k\,a)$, for some k satisfies $k.m = 1$. Hence I_a is the maximum.

Since the cardinal number of the set M_2^* is $p - 1$. Thus, there are $p - 1$ choices for $b \neq 0$ in (a, b). Including $(a, 0)$ and $(0, a)$ adds $p - 1$ more vertices. Adjusting for distinctness, the total is $(p - 1) + 2 = p + 1$. ■

Remark 3.2.

i. The set I_a is not unique; another maximum independent set exist, which is $\{(b, a): b \in M_2\} \cup \{(a, 0)\}$ in Γ_{2p},
ii. There are $2(p - 1)$ maximum independent sets in Γ_{2p}.
iii. The vertex $(0, a) \in I_a$ can be replaced with the vertex (p, a).
iv. All maximum independent sets of size $p + 1$ are isomorphic.

Example 3.2. Consider $n = 2 \times 7$, the set:

$$I_2 = \{(2,0),(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),(0,2)\}$$

is an independent set.

Proposition 3.7. The vertex connectivity of Γ_{2p}, is $\kappa(\Gamma_{2p}) = 4$.

Proof: The vertex $(0,0)$ is connected to all the vertices in Γ_{2p}. If we remove it, the remaining vertices form a connected subgraph. The vertices $\{(0,p),(p,0),(p,p)\}$ save the connectivity with the vertices in $M_2 \times M_2$. To isolate the vertices in $M_2 \times M_2$, we must remove all vertices $(0,0),(0,p),(p,0)$ and (p,p), as each component is presented by linear combinations of a vertex $(a,b) \in M_2 \times M_2$, so the resulting subgraph $\Gamma_{2p}/\{(0,0),(0,p),(p,0),(p,p)\}$ isomorphic to $(p-1)K_{p-1}$. Hence $\kappa(\Gamma_{2p}) = 4$.

Proposition 3.8. The graph Γ_{2p} contains only two distinct maximum cliques of size $2p$.

First Clique:

$$\mathcal{C}_L = \{(0,0),(2,0),(4,0),\dots,(2p-2,0),(0,p),(2,p),\dots,(2p-2,p)\}.$$

Second Clique:

$$\mathcal{C}_R = \{(0,0),(0,2),(0,4),\dots,(0,2p-2),(p,0),(p,2),\dots,(p,2p-2)\}.$$

Proof: Consider $\mathcal{C}_L$, from the definition of Γ_{2p}, it is clear that any two vertices $(a,0)$ and $(b,0)$ are adjacent. Also, any two vertices $(a,0)$ and (c,p) are adjacent. In addition, any two (c,p) and (d,p) are adjacent. The clique $\mathcal{C}_R$ is symmetric to $\mathcal{C}_L$, with coordinates swapped.

To prove the maximality, consider adding any vertex outside $\mathcal{C}_L$ (e.g., (p,p) breaks completeness: (p,p) is not adjacent to $(2,p)$ because $0 \not\equiv p^2 \pmod{n}$ fails.

The number of vertices in both cliques is:

$$|\mathcal{C}_R| = |\mathcal{C}_L| = \left(\frac{2p-2}{2} + 1\right) \times 2 = (p-1+1) \times 2 = 2p.$$

and the vertex degree is $2p - 1$. ■

Remark 3.3.

i. The maximum cliques $\mathcal{C}_R$ and $\mathcal{C}_L$ intersect only at $(0,0)$.

ii. Both cliques exploit the universal adjacency of $(0,0)$ and the zero-product property of $M_2 \cup M_p$.

Corollary 3.5 The clique number for the graph Γ_{2p} is:

$$\omega(\Gamma_{2p}) = 2p$$

Proposition 3.9. For an odd prime p, the graph Γ_{2p} is supereulerian.

Proof: To prove that Γ_{2p} is supereulerian, we construct a spanning closed trail.

The subgraphs induced by the subsets S_0 to S_6 exhibit the following properties:

i. The subgraph defined on S_1 forms a clique K_{p-1}, $S_1 \cup S_6$ and $S_3 \cup S_5$ form a larger clique $K_{2(p-1)}$.

ii. The subgraph defined on S_4 consists of $p-2$ disjoint cliques, each isomorphic to K_{p-1}, denoted as the following:

$$A_1 = \{(a_{1i}, b_{1i}) : a_{1i}, b_{1i} \in M_2^*, 1 \le i \le p-1\},$$

$$A_2 = \{(a_{2i}, b_{2i}) : a_{2i}, b_{2i} \in M_2^*, 1 \le i \le p-1\},$$

$$\vdots$$

$$A_{p-2} = \{(a_{(p-2)i}, b_{(p-2)i}) : a_{(p-2)i}, b_{(p-2)i} \in M_2^*, 1 \le i \le p-1\},$$

for each $j \in \{1,2,\dots,p-2\}$ and $a_{ji} \neq b_{ji}$ the subgraphs of Γ_{2p} on the sets A_j forms cliques K_{p-1}.

Since every complete graph contains a Hamiltonian path, denoted P_{S_i} and P_{A_i}, we can construct a spanning closed trail by concatenating paths from each subset as follows:

$$\{(0,0),P_{S_1}\} \cup P_{S_6} \cup \{(0,p),P_{S_2}\} \cup \{(p,0),P_{S_3}\} \cup P_{S_5} \cup \{(p,p),P_{A_1}\} \cup \{(0,0),P_{A_2}\} \cup \{(0,0),P_{A_3}\} \cup \dots \cup \{(0,0),P_{A_{p-2}},(0,0)\}$$

This trail is closed and spans all vertices of Γ_{2p}, proving the supereulerian property. ■

The closed trail in Proposition 3.9 results from systematically connecting paths across the grouped subsets S_0 to S_6. Figure 2 shows this process for Γ_{10} (where $p = 5$), demonstrating how Hamiltonian paths from cliques A_i and connector vertices like $(0,0)$ come together into a single cycle.

Corollary 3.6. For $p = 3$, the graph Γ_6 is Hamiltonian.

Proof: Using the same partition as above, the set S_4 in Γ_6 reduces to a single clique. The constructed closed trail simplifies to:

$$\{(0,0),P_{S_1}\} \cup P_{S_6} \cup \{(0,3),P_{S_2}\} \cup \{(3,0),P_{S_3}\} \cup P_{S_5} \cup \{(3,3),P_{S_4},(0,0)\},$$

Since all edges in this trail are distinct and every

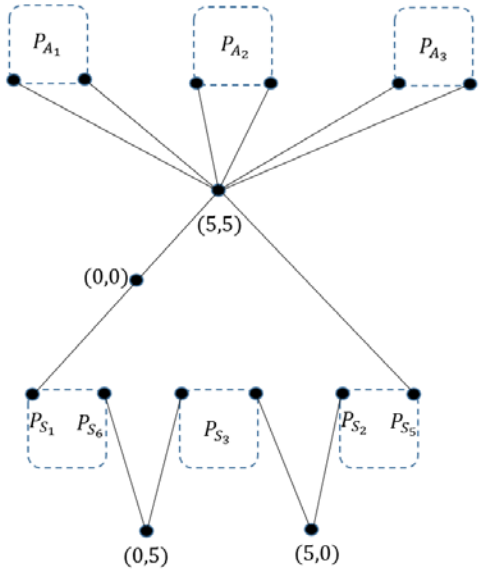

Figure 2: shown is closed spanning trail in Γ_{10}.

vertex is visited exactly once; the trail forms a Hamilton cycle. Thus, Γ_6 is Hamiltonian. ■

Case 2: if $n = p\,q$ is a composite.

Let $n = pq$, where p and q are distinct primes with $p > q$. The set of zero divisors in $\mathbb{Z}_{pq}$ decomposes into the union of two principal ideals:

$$\mathbb{D}_{pq} = M_p \cup M_q$$

where $M_q = \{\, kq : 0 \leq k \leq p-1\}$ is the maximal ideal generated by q, and $M_p = \{mp : 0 \leq m \leq q-1\}$ is the ideal generated by p. These sets exhibit an annihilation duality:

i. Every $x \in M_p$ satisfies $x \cdot q \equiv 0 \pmod n$, meaning $M_p \subseteq ann(q)$.
ii. Similarly, every $y \in M_q$ satisfies $y \cdot p \equiv 0 \pmod n$, so $M_q \subseteq ann(p)$.

This decomposition reflects the Chinese Remainder Theorem, as $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$, and the zero divisors arise precisely from the non-trivial multiples of p and q.

When $n = pq$ for distinct odd primes p, and q, the adjacency condition induces a rich structure reflecting the arithmetic of $\mathbb{Z}_n$. The vertex degrees in Γ_{pq} reveal fundamental properties of this graph, including its connectivity, symmetry, and relationship to zero-divisor interactions in $\mathbb{Z}_n \times \mathbb{Z}_n$.

Before we establish key propositions characterizing degrees of vertices, we partition the vertex set into subsets S_0 through S_{12}, as follows:

$S_0 = \{(0,0)\}$,
$S_1 = \{(a_i, 0) : a_i \in M_q^*, 1 \leq i \leq p-1\}$,
$S_2 = \{(0, a_i) : a_i \in M_q^*, 1 \leq i \leq p-1\}$,
$S_3 = \{(a_i, a_i) : a_i \in M_q^*, 1 \leq i \leq p-1\}$,
$S_4 = \{(a_i, pq - a_i) : a_i \in M_q^*, 1 \leq i \leq p-1\}$,
$S_5 = \{(a_i, b_i) : a_i, b_i \in M_q^*, 1 \leq i \leq p-1\}$,
$S_6 = \{(x_i, 0) : x_i \in M_p^*, 1 \leq i \leq q-1\}$,
$S_7 = \{(0, x_i) : x_i \in M_p^*, 1 \leq i \leq q-1\}$,
$S_8 = \{(x_i, x_i) : x_i \in M_p^*, 1 \leq i \leq q-1\}$,
$S_9 = \{(x_i, pq - x_i) : x_i \in M_p^*, 1 \leq i \leq q-1\}$,
$S_{10} = \{(x_i, y_i) : x_i, y_i \in M_p^*, 1 \leq i \leq q-1\}$,
$S_{11} = \{(a_i, y_j) : a_i \in M_q^*, y_i \in M_p^*, 1 \leq i \leq p-1, 1 \leq j \leq q-1\}$,
$S_{12} = \{(x_i, b_i) : x_i \in M_p^*, b_i \in M_q^*, 1 \leq i \leq q-1, 1 \leq j \leq p-1\}$,

The sizes of these subsets are:

$|S_1| = |S_2| = |S_3| = |S_4| = p - 1$,

$|S_5| = (p-1)(p-1) - 2(p-1) = p^2 - 4p + 3$,

$|S_6| = |S_7| = |S_8| = |S_9| = q - 1$,

$|S_{10}| = (q-1)(q-1) - 2(q-1) = q^2 - 4q + 3$

$|S_{11}| = |S_{12}| = (p-1)(q-1)$

Proposition 3.10. For $x, y \in M_p^*$, and $a, b \in M_q^*$, the degree of vertices in Γ_{pq} are classified as follows:

$$\deg(v) = \begin{cases} (p+q)^2 - 2(p+q) & if\ v = (0,0) \\ q^2 + (p-1)q - 1 & if\ v = (a,0)\ or\ (0,a) \\ pq - 1 & if\ v = (a,x)\ or\ (x,a) \\ p^2 + (q-1)p - 1 & if\ v = (x,0)\ or\ (0,x) \\ q^2 + p - 2 & if\ v = (a,b) \\ p^2 + q - 2 & if\ v = (x,y) \end{cases}$$

Proof: Since the vertex $v = (0,0)$ is adjacent to each single vertex in $V(\Gamma_{pq})$, and the adjacency relation is defined on different vertices, then

$\deg(v) = \big(n - \phi(n)\big)^2 - 1 = (p+q)^2 - 2(p+q)$.

Consider $v = (a,0)$, then v is adjacent to the vertices in $S_0, S_1, S_6, S_7, S_8, S_9, S_{10}$ and S_{11}. The first set S_0 represents a trivial annihilator, while the set S_1 represent vertices with linear dependence, and the sets S_6, S_7 and S_{10} represent vertices with annihilator entries. Since $|S_0| = 1$, $|S_1| = p - 2$ ($(a,0)$ is not counted), $|S_6| = |S_7| = |S_8| = |S_9| = q - 1$, $|S_{10}| = q^2 - 4q + 3$, and $|S_{11}| = (p-1)(q-1)$. Thus,

$$\begin{aligned} \deg(v) &= 1 + (p-2) + 4(q-1) + \\ &\quad (q^2 - 4q + 3) + (p-1)(q-1) \\ &= q^2 + (p-1)q - 1. \end{aligned}$$

In a similar manner, the degree of the vertex $v = (0,a)$ is $q^2 + (p-1)q - 1$.

If $v = (a,x)$, then v is adjacent to the vertices in S_0, vertices in S_1, and to the vertices with annihilator entries, i.e., vertices in S_7. Furthermore, vertices

with linear dependence, i.e., vertices in S_{11}. Since $|S_0| = 1$, $|S_1| = p-1$, $|S_7| = q-1$, and $|S_{11}| = (p-1)(q-1) - 1$ ((a,x) is not counted). Thus,

$$\deg(v) = 1 + (p-1)(q-1) - 1 + (q-1) + (p-1) = pq - 1.$$

Likewise, the degree of the vertex (x,a) is $pq-1$.

Let $v = (x,0)$, as usual, v is adjacent to S_0. Since $ann(p) = M_q$, then v is adjacent to vertices with annihilator entries, i.e., v is adjacent to vertices in S_3, S_4, and S_5. Moreover, it is adjacent to vertices with mixed entries, such as vertices in S_{12}. Also, v is adjacent to its linearly dependent vertices, i.e., vertices in S_6. Furthermore, v is adjacent to the vertices in the sets S_1 and S_2. Since $|S_0| = 1$, $|S_1| = |S_2| = |S_3| = |S_4| = p-1$, $|S_5| = p^2 - 4p + 3$, $|S_6| = q-2$ ($(x,0)$ is not counted), $|S_{12}| = (p-1)(q-1)$. Thus,

$$\deg(v) = 1 + 4(p-1) + p^2 - 4p + 3 + (q-2) + (p-1)(q-1) = p^2 + (q-1)p - 1$$

Similarly, the degree of the vertex $(0,x)$ is $p^2 + (q-1)p - 1$.

Let $v = (a,b)$ for any non-zeros $a, b \in M_q^*$. It is clear that (a,b) is adjacent S_0 and adjacent to the vertices in S_8, S_9 and S_{10}. The vertex v is adjacent to all linearly dependent vertices, i.e., vertices of the form (ma, mb), and they are p $-$ 2 vertices. Also, v is adjacent to the vertices in the sets S_6 and S_7. Typically, since $|S_0| = 1, |S_3| = |S_4| = p-2$ ($(a,$b$)$ is not counted), $|S_6| = |S_7| = |S_8| = |S_9| = q-1$, and $|S_{10}| = q^2 - 4q + 3$. Thus, the total degree of v is

$$\deg(v) = 1 + (p-1) + 4(q-1) + q^2 - 4q + 3 = q^2 + p - 2.$$

Finally, if $v = (x,y)$ for any non-zeros $x, y \in M_p^*$. It is obvious that (x,y) is adjacent to S_0 and adjacent to the vertices in the sets S_1, S_2, S_3, S_4 and S_5. Also, it is adjacent to linear dependence vertices (mx, my) for some $1 \le m \le q-1$. Since $|S_0| = 1$, $|S_1| = |S_2| = |S_3| = |S_4| = p-1$, $|S_5| = p^2 - 4p + 3$ and there are $q-2$ multiples of v. Thus, the total degree of v is:

$$\deg(v) = 1 + 4(p-1) + (p^2 - 4p + 3) + (q-2) = p^2 + q - 2.\blacksquare$$

Remark 3.4.

i. The degrees reflect the number of solutions to $xb \equiv ay \ (mod\ pq)$, tied to the Chinese Remainder Theorem.
ii. The structure generalizes to arbitrary n via prime factorization, with degrees computable multiplicatively.
iii. There is a spanning tree with exactly $(\phi(n))^2 - 1$ edges and $(\phi(n))^2$ vertices, which corresponds to: $\{(0,0),(a_1,b_1)\} \cup \{(0,0),(a_2,b_2)\} \cup \ldots \cup \{(0,0),(a_k,b_k)\}$, where $k = (n - \phi(n))^2 - 1$.
iv. The graph Γ_{pq} is not regular, where vertex degrees vary based on the linear dependence of their structure.

Corollary 3.7. The edge connectivity of Γ_{pq} is $\kappa'(\Gamma_{pq}) = pq - 1$.

Proposition 3.11. The cut vertex set of Γ_{pq} is the ideal-generated subgraph:
$V_c = M_p \times M_p = \{(kp, lp): k, l \in \{0,1,\ldots,q-1\}$, where $M_p = \langle p \rangle \subset \mathbb{Z}_{pq}$.

Proof: By definition, $M_p \times M_p$ includes all pairs where both coordinates are multiples of p (zero divisors modulo q). From Proposition 3.10, the vertices in V_c have degree $p^2 + (q-1)p - 1$ or higher.

The subgraph $M_q \times M_q$, where both coordinates are multiples of q, becomes disconnected. For (a,b), $(x,y) \in M_q \times M_q$, the adjacency condition implies that q^2 divides $(ay - bx)$, which only holds if $(a,b) = k(x,y)$. Without V_c, there are no paths between different proportional pairs. Pairs like (a,x) with $a \in M_q$, $x \in M_p$ lose all connections to $M_q \times M_q$ when V_c is removed, because V_c previously bridged them via $(0,0)$ and $(0,x)$.

Observe that no proper subset of V_c suffices keeping $(0,0)$ preserves some paths, but $M_p \times M_p \backslash \{(0,0)\}$ still disconnects $M_q \times M_q$ due to the constraints of the Chinese Remainder Theorem. Smaller sets fail to separate all $q-1$ equivalence classes in $M_q \times M_q$. ■

Corollary 3.8. The vertex connectivity of the graph Γ_{pq} is $\alpha(\Gamma_{pq}) = p^2$, achieved by the minimal cut set $V_c = M_p \times M_p$.

Example 3.3. For $n = 15$, where $q = 3, p = 5$:
$V_c = M_5 \times M_5 = \{(0,0),(5,0),(10,0),(0,5),(0,10),(5,5),(10,10),(10,5),(5,10)\}$.
Removing V_c leaves 4 isolated cliques from $M_3 \times M_3 = \{(3,0),\ldots,(14,14)\}$ since $(0,0) \in V_c$ has already been removed. There are also two larger cliques with mixed coordinates. See Figure 3.

Proposition 3.12. The graph Γ_{pq} contains two distinct types of maximum cliques, based on the graph definition on $\mathbb{D}_n \times \mathbb{D}_n$:

Left clique

$$\mathcal{C}_l = \{(k\ p, a): 0 \le k \le q-1, a \in M_q\}.$$

Right clique

$$\mathcal{C}_R = \{(a, k\ p): 0 \le k \le q-1, a \in M_q\}.$$

Proof: For any two vertices $(k_1\ p, a_1), (k_2\ p, a_2) \in \mathcal{C}_l$, their adjacency condition is:

$(k_1\ p)\ (a_2) - (a_1)(k_2\ p) \equiv p(k_1\ a_2 - k_2\ a_1)$, as $k_1\ a_2 - k_2\ a_1 \in M_q$, then $p(k_1\ a_2 - k_2\ a_1) \equiv 0\ (mod\ pq)$, which implies $k_1\ a_2 - k_2\ a_1 \equiv 0\ (mod\ q)$.

equivalence classes of linearly dependent and annihilator coordinator vectors over $\mathbb{D}_n \times \mathbb{D}_n$.

In the graph Γ_{15} Figure 4 illustrates the maximum left clique including smaller left cliques.

Corollary 3.9. In the graph Γ_{pq}, the clique number $\omega(\Gamma_{pq}) = pq$

Proposition 3.13. Let $n = pq$ for distinct odd primes p, q. The set

$$I_a = \{(a, m\ a): m = 0, 1, 2, \dots, p-1, a \in M_q^*\} \cup \{(x, b)\},$$

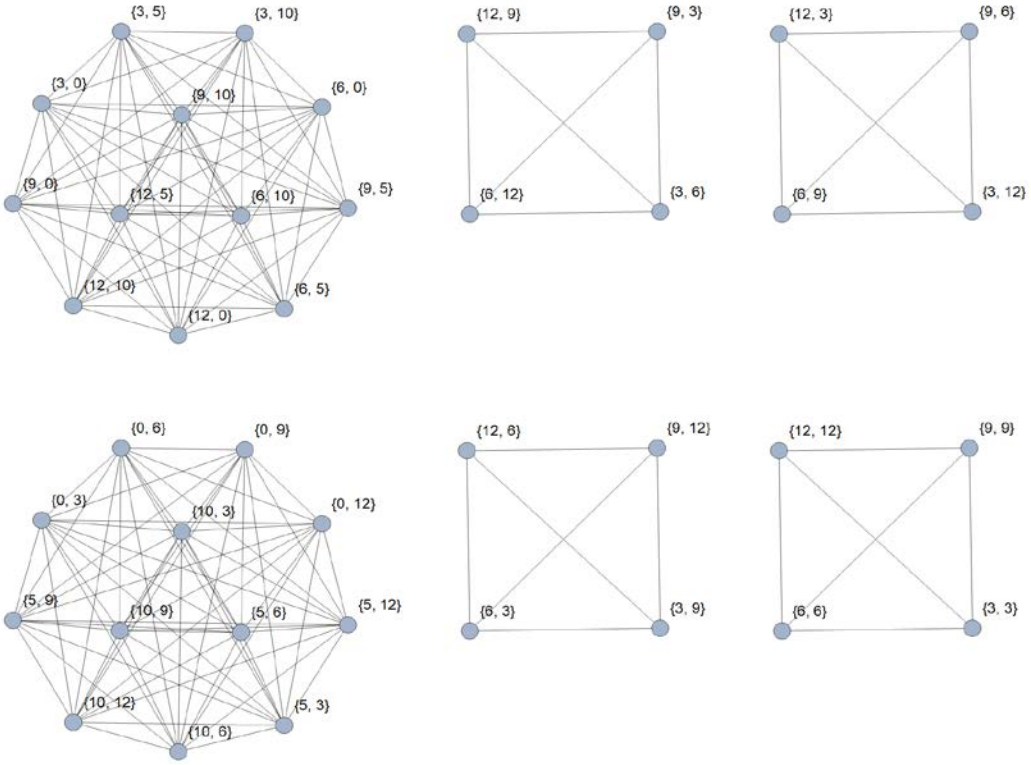

Figure 3: The graph Γ_{15} without vertex cut set V_c

Moreover, this is the maximum clique (no proper superset is a clique) because extending any external vertex (x, y) with $y \not\equiv 0\ (mod\ q)$ and $y \in M_p$ to $\mathcal{C}_l$ breaks adjacency for some pairs, where $(k\ p)\ y \not\equiv 0\ (mod\ pq)$. Similarly, $\mathcal{C}_R$ is a right maximum clique. ■

Remark 3.5.

i. The intersection of any left and right clique is exactly {(0,0)}.

ii. There are more smaller cliques, such as $\{(0,0), (0, a_1), (0, a_2), \dots, (0, a_{p-1})\}$ and $\{(0,0), (x_1, 0), (x_2, 0), \dots, (x_{q-1}, 0)\}$, where $m = pq - \phi(pq) + 1 = p + q$.

iii. If the vertex cut set is removed from Γ_{pq}, the resulting graph decomposes into a union of complete subgraphs (cliques) corresponding to

where $x \in M_p$ and $b \in M_q^*$, is a maximum independent set in Γ_{pq} with $\alpha(\Gamma_{pq}) = p + 1$.

Proof: Let $v_1 = (a, m_1 a)$ and $v_2 = (a, m_2 a)$ be any two vertices in I_a. Based on the adjacency condition, then $a(m_2\ a) - (m_1\ a)a \equiv a^2(m_2 - m_1)$. Since $m_2 - m_1 \neq 0$ and $a^2 \not\equiv 0\ (mod\ pq)$. Thus, the condition fails. If $v_2 = (x, b)$ then $ab - (m_1\ a)x \equiv ab \not\equiv 0\ (mod\ n)$, which means I_a is an independent vertex set.

To prove the maximality, suppose we add any other vertex $(x, y) \in \mathbb{D}_n \times \mathbb{D}_n$ to I_a. Then, If $x \in M_q$, then (x, y) is adjacent to $(a, 0) \in I_a$ (since $x \cdot 0 - a \cdot y \equiv 0\ (mod\ pq)$ if and only if $y \equiv 0\ (mod\ p)$, but y must be a multiple of a to avoid adjacency with $(a, 0)$, leading to $(x, y) \in I_a$.

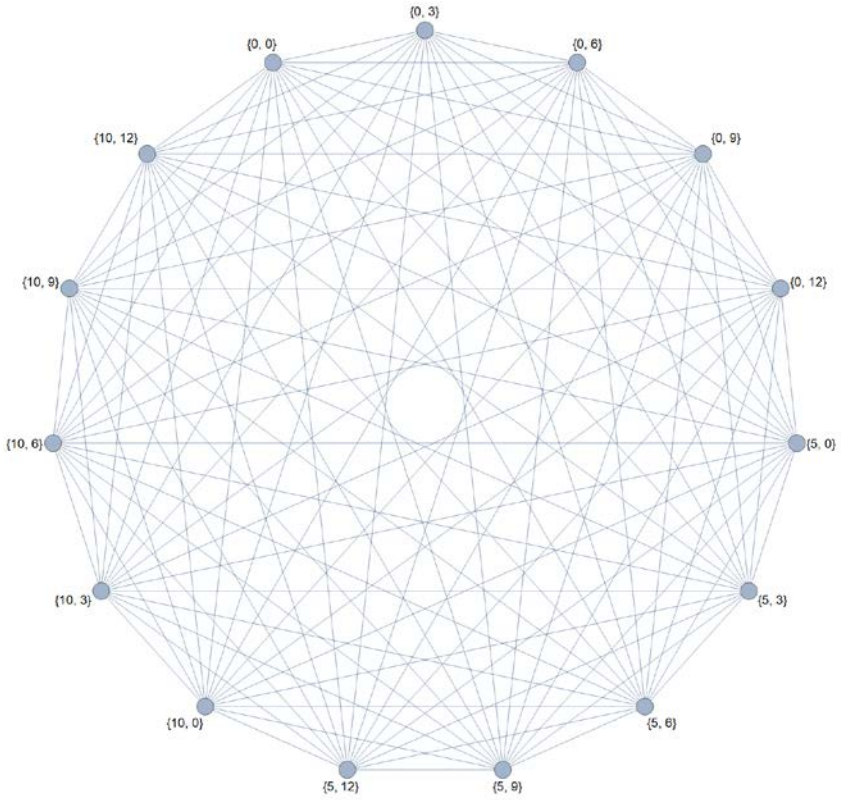

Figure 4: The maximum left clique in Γ_{15}

On the other side, if $x \in M_p$, then (x, y) is adjacent to $(\beta, b) \in I_a$ for some $\beta \in M_p$ (since $x\ b - \beta\ y \equiv 0\ (mod\ pq)$ if and only if $y \equiv\ 0\ (mod\ q)$, but y must be a multiple of p to avoid adjacency with (β, b), leading to (x, y) is adjacent to $(a, m\ a)$.

Therefore, the set I_a cannot be extended with a vertex $(x, y) \in M_p^* \times M_p^*$ without violating independence. Hence I_a is a maximum independent set in Γ_{pq}.

To compute the $\alpha(\Gamma_{pq})$, the subset $\{(q, mq): 0 \leq m \leq p-1\}$ contains exactly p distinct vertices. Adding (α, a), which is distinct from all vertices in I_a gives $p+1$ total vertices. ■

Proposition 3.14. For odd primes p and q with $p > q$ if the vertex connectivity $\kappa(\Gamma_{pq}) \geq p+1$, then the graph Γ_{pq} is Hamiltonian.

Proof: To determine the Hamiltonicity of Γ_{pq}, we combine our analysis of the graph's partition into sets S_0 through S_{12} (Section 3.2) with the classical sufficient condition of Chvátal and Erdős (Theorem 2.2). This condition states that a graph is Hamiltonian if its vertex connectivity $\kappa(G)$ is at least its independence number $\alpha(G)$. From Proposition 3.13, we know that $\alpha(\Gamma_{pq}) = p+1$, and according to this proposition, $\kappa(\Gamma_{pq}) \geq\ p+1$. Therefore, the Chvátal-Erdős condition holds as long as Γ_{pq} is connected, which we will confirm through explicit construction.

To construct a Hamiltonian cycle, we exploit the graph's partitioned structure (section 3.2) and connectivity properties.

The subgraph defined on the set S_5 contains $p-3$ components (cliques), while the set S_{10} contains $q-3$ components (cliques). We denote these components as:

$$A_1 = \{(a_{1i}, b_{1i}): a_{1i}, b_{1i} \in M_q^*, 1 \leq i \leq p-1\},$$
$$A_2 = \{(a_{2i}, b_{2i}): a_{2i}, b_{2i} \in M_q^*, 1 \leq i \leq p-1\},$$
$$\vdots$$
$$A_{q-3} = \{(a_{(q-3)i}, b_{(q-3)i}): a_{(q-3)i}, b_{(q-3)i} \in M_q^*, 1 \leq i \leq p-1\}$$

for S_5, and

$$X_1 = \{(x_{1i}, y_{1i}): x_{1i}, y_{1i} \in M_p^*, 1 \leq i \leq q-1\}$$
$$X_2 = \{(x_{2i}, y_{2i}): x_{2i}, y_{2i} \in M_p^*, 1 \leq i \leq q-1\},$$
$$\vdots$$
$$X_{q-3} = \{(x_{(q-3)i}, y_{(q-3)i}): x_{(q-3)i}, y_{(q-3)i} \in M_p^*, 1 \leq i \leq q-1\},$$

for S_{10}.

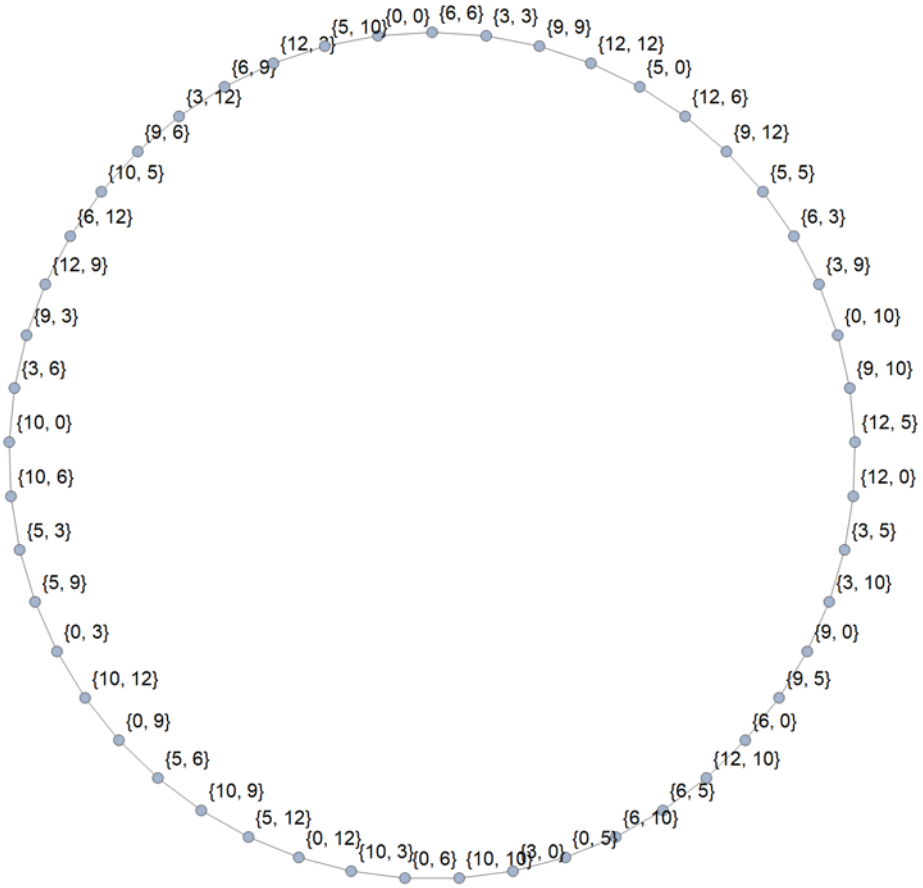

Figure 5: For $p = 5$ and $q = 3$, shown is the Hamiltonian cycle of the graph Γ_{15}

Each is constructed to preserve specific algebraic relationships. These subsets form complete subgraphs, ensuring the existence of Hamiltonian paths within them.

Key subsets include S_5 and S_{10}, which decompose into smaller complete subgraphs $A_1, \dots, A_{p-3}$ and $X_1, \dots, X_{q-3}$, respectively. Each of these components supports a Hamiltonian path, denoted P_{A_i} and P_{X_i}. The proof proceeds by systematically connecting these paths: P_{S_4} links to P_{S_7}, which then connects to $P_{S_{11}}$, and this chain continues through P_{S_1},(0,0), P_{S_2}, $P_{S_{12}}$ and P_{S_6} ensuring all vertices are traversed without repetition.

To connect more paths, P_{S_6} must be connected to a path P_{A_i} for $1 \le i \le p-3$, then a path P_{X_j} for $1 \le j \le q-3$. Since $p-3$ is greater than $q-3$, we supplement more vertices excluded from P_{S_6}, P_{S_7}, P_{S_8} and P_{S_9}, ensuring that all of them are minimized to smaller paths up to at least one vertex remains to preserve connectivity between Hamiltonian paths. This supplement allows us to connect the rest of the paths P_{A_i}, and this creation of paths ends with a path P_{A_i}, which can be connected to the path P_{S_8}, P_{S_3} then P_{S_9}. This flexibility guarantees that the construction remains valid even when P_{S_9} and $P_{S_{10}}$ lack enough connectors.

The final connection loops back P_{S_9} to P_{S_4}, completing the Hamiltonian cycle. However, if we subtract three vertices from the cut vertex set, which are $\{(0,0),(0,x_j),(x_j,0)\}$, we get the number of vertices required to connect the paths A_i, P_{S_3} and $\kappa(\Gamma_{pq}) - 3 = q^2 - 3 \ge p - 2$. Therefore, the graph G_{pq} admits a Hamiltonian cycle if $\kappa(\Gamma_{pq}) \ge p + 1$. ■

Example 3.4. Consider $n = 5 \times 3$, then $\kappa(\Gamma_{15}) = 9$, and $p + 1 = 6$. It is Hamiltonian. See Figure 5. However, if $n = 11 \times 3$, then $\kappa(\Gamma_{33}) = 9$, and $p + 1 = 12$. So, Γ_{33} isn't Hamiltonian, because the Hamilton paths such as P_{S_4} and P_{A_8} are not adjacent. Based on the sufficient condition for an undirected graph to be supereulerian, provided by Theorem 2.3, we have the following:

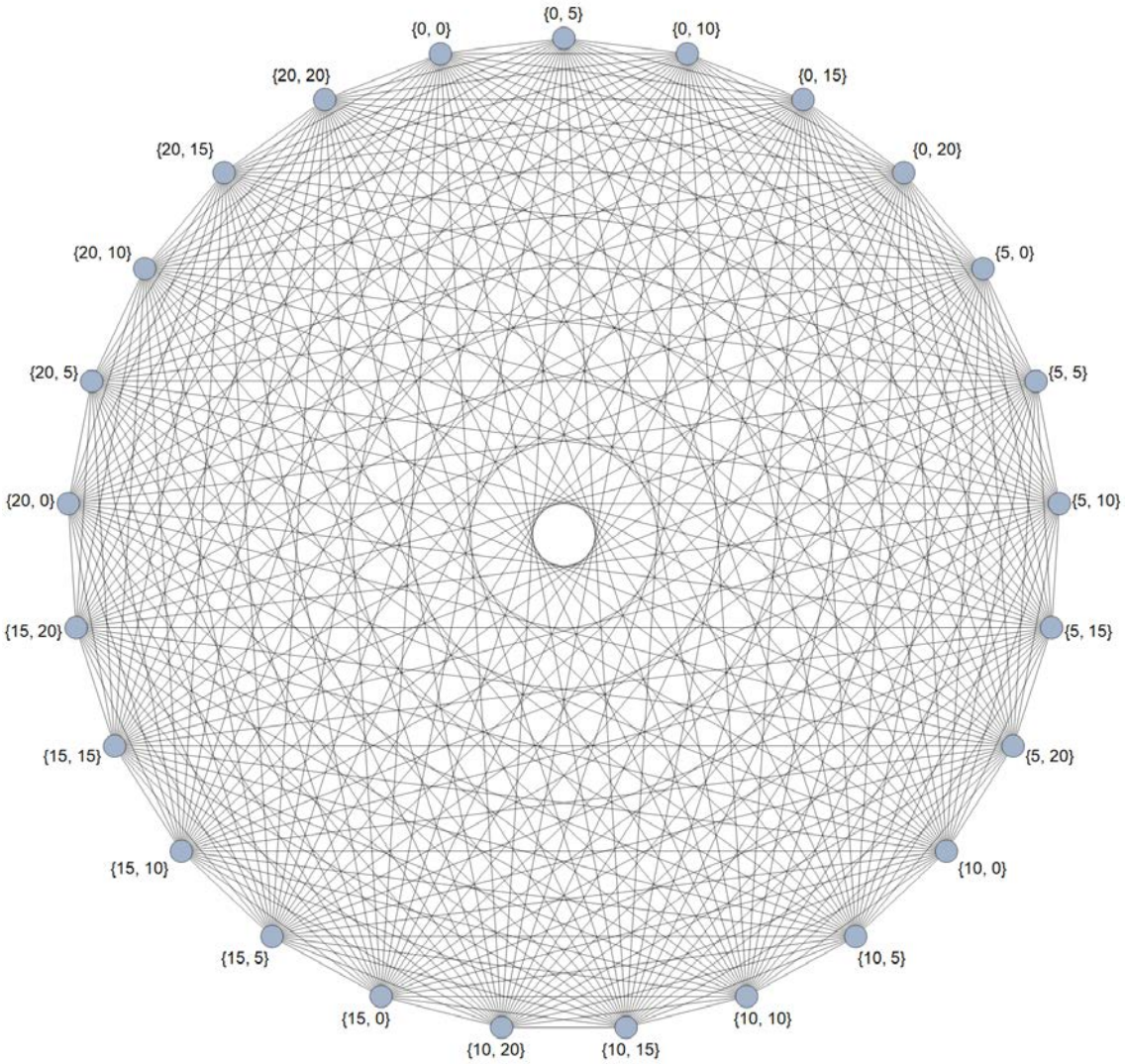

Figure 6: Shown is complete graph Γ_{25}, for $p = 5$.

Corollary 3.10. If $\kappa(\Gamma_{pq}) \geq p + 1$. Then Γ_{pq} is supereulerian.

Proof: A graph is supereulerian if it contains a spanning closed trail. By Proposition 3.14, Γ_{pq} with $\kappa(\Gamma_{pq}) \geq p + 1$ is Hamiltonian, then it has a Hamiltonian cycle C_H that visits every vertex exactly once. Since C_H is a closed trail spanning all vertices, Γ_{pq} is trivially supereulerian.

Remark 3.6. The converse is false; supereulerian graphs need not be Hamiltonian (e.g., the union of two cycles sharing one vertex is supereulerian but not Hamiltonian).

Corollary 3.11. For odd primes p and q with $p = q + 2$, the graph Γ_{pq} is Hamiltonian.

Proof: All vertices in the following Hamiltonian path are included exactly once, as each subset's Hamiltonian sub-path covers its vertices, and the linking process preserves this property. Thus, the constructed cycle is Hamiltonian.

$$\{P_{A_1}, (x_1, x_1)\} \cup \{P_{A_2}, (x_2, x_2)\} \cup \ldots \cup \{P_{A_{q-1}}, (x_{q-1}, x_{q-1})\} \cup \{(a_1, a_1), P_{X_1}\} \cup \{(a_2, a_2), P_{X_2}\} \cup \ldots \cup \{(a_{q-3}\ a_{q-3}), P_{X_{q-3}}\} \cup \{(a_{q-2}, a_{q-2}), (a_{q-1}, a_{q-1}), (a_q, a_q), (a_{q+1}, a_{q+1})\} \cup P_{S_9} \cup P_{S_4} \cup P_{S_7} \cup P_{S_{11}} \cup \{P_{S_1}, (0,0)\} \cup P_{S_2} \cup P_{S_{12}} \cup P_{S_6} \cup P_{A_1}.$$ ■

Corollary 3.12. For odd primes p and q with $p = q + 2$, the graph Γ_{pq} is supereulerian.

Case 3: If $n = p^2$ for an odd prime p.

When $n = p^2$ where p is an odd prime. In Γ_{p^2} every creates the zero-divisor set:

$$\mathbb{D}_{p^2} = \{\, kp : 0 \leq k \leq p - 1 \,\}$$

The annihilator of any zero divisor $k\,p$ is the principal ideal $< p >$.

For any two vertices $(a,b) = (k_1p, k_2p)$ and $(x,y) = (k_3p, k_4p)$ in $\mathbb{D}_{p^2} \times \mathbb{D}_{p^2}$, the bilinear form simplifies to:

$$ay - bx = (k_1k_4 - k_2k_3)p^2 \equiv 0\ (mod\ p^2).$$

This congruence holds for all $k_i \in \mathbb{Z}_p$ because p^2 always divides $(k_1k_4 - k_2k_3)p^2$. Therefore, every pair of distinct vertices in Γ_{p^2} is adjacent, making Γ_{p^2} isomorphic to the complete graph K_{p^2}.

This simplicity sharply differs from the situation where $n = pq$ with distinct primes. In that case, non-adjacent pairs appear. For instance, $(p,0)$ and $(0,p)$ are not adjacent.on-unit element of $\mathbb{Z}_{p^2}$ is a multiple of p, which

Proposition 3.15. For any odd prime number p, the graph Γ_{p^2} is complete

Proof: Completeness follows from the universal vanishing of bilinear congruence, while uniqueness holds because for $n = pq$ (distinct primes), the condition $ay \equiv bx\ (mod\ pq)$ fails for pairs like $(p,0)$ and $(0,q)$. ■

The complete structure of Γ_{p^2} from Proposition 3.15 is shown in Figure 6. This figure illustrates Γ_{p^2} for $p = 5$. All possible edges are included, confirming the universal adjacency of zero-divisor pairs (kp, lp) in $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.

When $n = p^2$, zero-divisor dominance leads the graph Γ_{p^2} to become a single maximal clique. Thus, $\omega(\Gamma_{p^2}) = p^2$.

Remark 3.7.

i. Unlike composite $n = pq$, zero-divisors in $\mathbb{Z}_{p^2}$ have identical annihilators $\langle\ p\ \rangle$, making all pairs adjacent.
ii. While H_{p^2} decomposes due to multiplicative structure, Γ_{p^2} collapses into a single clique from additive uniformity.
iii. The diameter of the graph Γ_{p^2} is $diam(\Gamma_{p^2}) = 1$ as Γ_{p^2} is complete, and the girth is $girth(\Gamma_{p^2}) = 3$.

Corollary 3.13. The graph is Hamiltonian and supereulerian.

Our study of G_n in these cases leads to a shared understanding: the prime factorization of n determines the overall structure of the graph. We summarize these findings below and explore their larger implications.

4. Conclusion

In this work, we have studied the graph G_n defined on $\mathbb{Z}_n \times \mathbb{Z}_n$ using the adjacency condition $ay \equiv bx\ (mod\ n)$. By breaking down G_n into unit and zero-divisor subgraphs, we uncovered a clear contrast. The unit subgraph H_n consists of separate cliques determined by Euler's totient function $\phi(n)$. In contrast, the zero-divisor subgraph Γ_z shows the complex interactions of annihilation ideals and prime factorization. Our study of the cases $n = p^2$, $n = 2p$, and $n = pq$ showed how the algebraic structure of $\mathbb{Z}_n$ directly influences the graph's shape. This ranges from the complete graph K_{p^2} for $n = p^2$ to the Hamiltonian and supereulerian properties appearing under certain connectivity conditions for $n = pq$.

List of symbols

Symbol	Description
$\mathcal{B}$	The bilinear form function
$\mathbb{Z}_n$	The ring of integers modulo n
$\mathbb{U}_n$	The group of units in $\mathbb{Z}_n$
$\mathbb{D}_n$	The set of zero divisors (including zero)
$ann(a)$	The annihilator of a
$\phi(n)$	Euler's totient function
M_2	The finite set of multiples of 2
M_p	The finite set of multiples of p
M_p^*	The finite set of multiples of p without zero
$\#(A)$	The cardinality of the set A.
S_i	A partition set of $\mathbb{D}_n \times \mathbb{D}_n$ for $0 \le i \le 12$
A_i	A partition set of S_5 for $n = pq$.
X_i	A partition set of S_{10} for $n = pq$.
G_n	The whole graph is defined on the ring $\mathbb{Z}_n \times \mathbb{Z}_n$ by the bilinear form $\mathcal{B}$
H_n	The subgraph defined on $\mathbb{U}_n \times \mathbb{U}_n$
Γ_n	The subgraph defined on $\mathbb{D}_n \times \mathbb{D}_n$
K_n	Complete graph with n vertices
$diam(G)$	The diameter of a graph G
$girth(G)$	The girth of a graph G
$\omega(G)$	The clique number of a graph G
deg (v)	Degree of a vertex v
$\kappa(\Gamma_n)$	Vertex Connectivity of a graph Γ_n
I_v	Independent vertex set
$\mathcal{C}_L$	The left clique
$\mathcal{C}_R$	The right clique
$(a,b) \sim (x,y)$	Shows the adjacency between the vertices $(a,b),(x,y)$
$\kappa'(\Gamma_n)$	Edge connectivity
P_{S_i}	A Hamilton path of the components S_i
P_{A_i}	A Hamilton path of the components A_i
P_{X_i}	A Hamilton path of the components X_i
$\alpha(\Gamma_n)$	The size of a maximal independent set in Γ_n
$\delta(G)$	The Minimum degree of a graph G.

References

[1]. D. F. Anderson and P. S. Livingston, "The Zero-Divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, no. 2, pp. 434–447, Jul. 1999, doi: 10.1006/jabr.1998.7840.

[2]. A. Badawi, "On the Dot Product Graph of a Commutative Ring," *Communications in Algebra*, vol. 43, no. 1, pp. 43–50, Aug. 2014, doi: 10.1080/00927872.2014.897188.

[3]. A. E. Brouwer and W. H. Haemers, *Spectra of graphs*. 2011. doi: 10.1007/978-1-4614-1939-6.

[4]. J. Rotman, *Advanced modern algebra*. 3rd ed., 2015. doi: 10.1090/gsm/165.

[5]. D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed., Wiley 1999.

[6]. P. Aluffi, *Algebra: Notes from the Underground*. Cambridge University Press, 2021.

[7]. D. Eisenbud, *Commutative algebra: with a View Toward Algebraic Geometry*. Springer Science & Business Media, 2013.

[8]. J. Gallian, *Contemporary Abstract algebra*. CRC Press, 2021.

[9]. C. Godsil and G. F. Royle, *Algebraic graph Theory*. Springer Science & Business Media, 2013.

[10]. G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, 7th ed. CRC Press, 2004.

[11]. D. B. West, *Introduction to graph Theory*. 2nd ed. Prentice Hall, 2001.

[12]. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*. London : Macmillan Press, 1976.

[13]. L. Han, H.-J. Lai, L. Xiong, and H. Yan, "The Chvátal–Erdös condition for supereulerian graphs and the Hamiltonian index," *Discrete Mathematics*, vol. 310, no. 15–16, pp. 2082–2090, Apr. 2010, doi: 10.1016/j.disc.2010.03.020.

[14]. J. Bang-Jensen and A. Maddaloni, "Sufficient Conditions for a Digraph to be Supereulerian," *Journal of Graph Theory*, vol. 79, no. 1, pp. 8–20, Jun. 2014, doi: 10.1002/jgt.21810.

[15]. P. A. Catlin, "Supereulerian graphs: A survey," *Journal of Graph Theory*, vol. 16, no. 2, pp. 177–196, Jun. 1992, doi: 10.1002/jgt.3190160209.

[16]. P. A. Catlin and H.-J. Lai, "SuperEulerian Graphs and the Petersen Graph," *Journal of Combinatorial Theory Series B*, vol. 66, no. 1, pp. 123–139, Jan. 1996. doi: 10.1006/jctb.1996.0009.

[17]. B. Cooperstein, *Advanced Linear Algebra*, 2nd ed. Chapman and Hall/CRC, 2015.