مجلة جامعة الزاوية للعلوم الطبيعية
University of Zawia Journal of Natural Sciences *(UZJNS)*
https://journals.zu.edu.ly/index.php/UZJNS ISSN: 3078-4999

UZJNS

Spanning Directed Trails in Semicomplete 3-Multipartite Digraphs

Omaema A Lasfar [a *]

[A]*Department of Mathematics, Faculty of Education, University of Zawia, Zawia, Libya.*
**Corresponding author:*

E-mail address: o.lasfar@zu.edu.ly

Received 24 Oct 2024 | Accepted 18 Feb 2025 | Available online 15 Mar 2025 | DOI: 10.26629/uzjns.2025.03

A B S T R A C T
The study of spanning trails in digraphs associates two fundamental concepts: connectivity and independence. For general digraphs, establishing universal conditions remains an open challenge, but semicomplete multipartite digraphs reveal precise connectivity restrictions due to their well-defined structure. In this work, we prove that every strongly connected semicomplete 3-multipartite digraph D with arc-strong connectivity $\lambda(D) \geq \alpha(D) - 1$ contains a spanning directed trail. This result provides evidence that multipartite structures may require weaker connectivity conditions than arbitrary digraphs, opening new directions for studying spanning trails in constrained digraph classes.

Keywords: arcs-strong connectivity; stability number; spanning trail; supereulerian (closed spanning trails) digraphs; semicomplete 3-multipartite digraphs.

المسار المتجهي الممتد في الرسم البياني الموجه الشبه المكتمل ذو ثلاث مجموعات جزئية

أميمة الأصفر [a]

كلية التربية، جامعة الزاوية، الزاوية، ليبيا

الملخص
تربط دراسة المسارات الشاملة في الرسوم البيانية الموجهة بين مفهومين أساسيين: الاتصال والاستقلالية. في حين أن وضع شروط عامة للرسوم البيانية الموجهة يظل تحديًا مفتوحًا، فإن الرسوم البيانية متعددة الأجزاء شبه المكتملة تكشف عن قيود اتصالية دقيقة بفضل بنيتها المحددة جيدًا. في هذا البحث، نثبت أن كل رسم بياني موجه شبه المكتمل متعدد الأجزاء من الدرجة الثالثة D ذو ترابط قوسي قوي $\lambda(D) \geq \alpha(D) - 1$ يحتوي على مسار موجه ممتد. توفر هذه النتيجة دليلًا على أن البنى متعددة الأجزاء قد تتطلب شروط اتصال أضعف من الرسوم البيانية العشوائية، مما يفتح آفاقًا جديدة لدراسة المسارات الشاملة في فئات مقيدة من الرسوم البيانية الموجهة.

الكلمات المفتاحية: *الترابط القوسي القوي، الرقم المستقل، المسار الممتد، الرسم البياني ذو المسار الممتد المغلق، الرسم البياني شبه المكتمل ذو ثلاث مجموعات جزئية.*

1. Introduction

We consider finite graphs and digraphs, and generally use G to denote a graph and D to a digraph. Undefined terms and notation will follow [1] for graphs and [2] for digraphs. Following [2], a digraph D does not have loops and parallel arcs, let $\lambda(D)$ and $\alpha(D)$ denote the arc-strong connectivity of a digraph D, and the stability number (also called the independence number), respectively. We shall use the term independent set for a stable set in a digraph.

A strong digraph D is eulerian, for any $v \in V(D), d_D^-(v) = d_D^+(v)$. A digraph D is supereulerian if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. Thus, supereulerian digraphs must be strong.

The supereulerian digraph problem is to characterize the strong digraphs that contain a spanning closed directed trail. Several studies on supereulerian digraphs have been conducted. In particular, Hong et al in [3], [4] and Bang-Jensen and Maddaloni [5] presented some best possible sufficient degree conditions for supereulerian digraphs. Several studies on various conditions of supereulerian digraphs can be found in [6], [7], [8], among others.

In 1972, Chvátal and Erdös [9] proved that every 2-connected graph G with $\kappa(G) \geq \alpha(G)$ is Hamiltonian. Thomassen [10] indicated that the Chvátal-Erdös Theorem does not extend to digraphs by presenting an infinite family of non-hamiltonian (but supereulerian) digraphs D with $\kappa(D) = \alpha(D) = 2$. This motivates Bang-Jensen and Thommassé to make the following conjecture.

Conjecture 1.1. (Bang-Jensen and Thommassé [5]) Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.

In [5], Bang-Jensen and Maddaloni studied the validity of Conjecture 1.1 for several families of digraphs one of them is the following theorem.

Theorem 1.2. Let D be a strong semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then D is superelerian. This is equivalent to saying that D has a spanning closed trail.

In addition, Bang-Jensen and Maddaloni [5] provided infinite families of digraphs with $\lambda(D) = \alpha(D) - 1$ that are not supereulerian. Hence, if true, Conjecture 1.1 would be best possible.

The current study is motivated by Conjecture 1.1 and Theorem 1.2, for a semicomplete 3-multipartite digraph. The main result of this note is the following.

Theorem 1.3. Let D be a strong semicomplete 3-multipartite digraph. if $\lambda(D) \geq \alpha(D) - 1$, then D has a spanning trail.

2. Preliminaries

Throughout our discussions, we use the notation (u, v) to denote an arc oriented from u to v in a digraph D; and use $[u, v]$ to indicate either (u, v) or (v, u). When $[u, v] \in A(D)$, we say that u and v are adjacent. If two arcs of D have a common vertex, we say that these two arcs are adjacent in D. The in-degree and the out-degree of a vertex v in a digraph D are denoted by $d_D^-(v)$ and $d_D^+(v)$, respectively. Throughout this paper, we use the definitions of paths, cycles, and trails as defined in [1] when discussing an undirected graph G, and we denote directed paths, directed cycles and directed trails when discussing it on a digraph D. A walk in D is an alternating sequence, $x_1 a_1 x_2 a_2 \dots x_{k-1} a_{k-1} x_k$ of vertices x_i and arcs a_i from D such that $a_i = x_i x_{i+1}$, for $i = 1, \dots, k-1$. A walk is closed if $x_1 = x_k$, and open otherwise. If all the arcs of a walk are distinct, we call it a trail. A directed trail (or path, respectively) from a vertex u to a vertex v in a digraph D is often referred to as a (u, v) −trail (a (u, v) −path, respectively). As in [1], we define, for a vertex $v \in V(D)$, $N_D^+(v) = \{w \in V(D) : (v, w) \in A(D)\}$, $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$, and $N_D(v) = N_D^+(v) \cup N_D^-(v)$.

For subsets $X, Y \subseteq V(D)$, define $(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}$,
and $(X, Y)_{G(D)} = (X, Y)_D \cup (Y, X)_D$.

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x, Y)_D$ for $(X, Y)_D$ or $(X, y)_D$ for $(X, Y)_D$, respectively. Hence $(X, Y)_D = (\{x\}, \{y\})_D$.

For a vertex $v \in V(D)$, let $\partial_D^+(v) = (v, V(D) - v)_D$ and $\partial_D^-(v) = (V(D) - v, v)_D$. Thus $d_D^+(v) = |\partial_D^+(v)|$ and $d_D^-(v) = |\partial_D^-(v)|$ and $d_D(v) = d_D^+(v) + d_D^-(v)$.

We use the definition of union digraphs as in [2]. By definition of $\lambda(D)$ in [2], we observe that for any integer $k \geq 0$ and a digraph D,

$\lambda(D) \geq k$ if and only if $|\partial_D^+(X)| \geq k$, for any nonempty proper subset $X \subset V(D)$.

3. Proof of the main result

Lemma 3.1. Suppose D is a semicomplete 3-multipartite digraph D with $\lambda(D) \geq \alpha(D) - 1 \geq 2$. Let $x, y \in V(D)$ such that $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1,2,3\}$ where $i \neq j$. Then either $(x, y) \in A(D)$ or there exists a vertex $v \in V(D)$ such that $(x, v), (v, y) \in A(D)$.

Proof: Assume V_1, V_2, and V_3 are the partition sets of a digraph D. Suppose $x \in V_1$ and $y \in V_2$ By the definition of D, x and y are adjacent. Assume $(x, y) \notin A(D)$, so $(y, x) \in A(D)$. By a definition of D, for each vertex $v \in V_3$, $[x, v], [y, v] \in A(D)$. If there exists $v \in V_3$ such that $(x, v), (v, y) \in A(D)$ then we are done. By contradiction, we assume that for all $v \in V_3$ with $(x, v) \in A(D)$ that $(y, v) \in A(D)$.

Define $S = \{u \in V_3 : u \in N_D^+(x) \cap N_D^+(y)\}$.

If $S = \emptyset$, then for each $v \in V_3$, either $v \in N_D^-(x) \cap N_D^-(y)$, or $v \in N_D^-(x) \cap N_D^+(y)$, it follows $|(x, V_3)_D| = 0$, which contracts the strong connectivity of D. We may assume $|S| \geq 1$. By assumption, $|S \cap N_D^-(y)| = 0$, thus $|(V_3, y)_D| \leq |V_3| - |S| \leq \alpha(D) - 1$, which contracts the arc-strong connectivity of D, so there must be a vertex $v \in V_3$ such that $(x, v), (v, y) \in A(D)$. ■

Lemma 3.2. Suppose D is a strong semicomplete 3-multipartite digraph D. Let $x, y \in V(D)$ such that $x, y \in V_i$ for some $i \in \{1,2,3\}$. Then, one of the following holds:

(a)There exists a vertex $v \in V(D)$ such that

$(x, v), (v, y) \in A(D)$.

(b)There is an (x, y) −path of length 3 which is the shortest one.

Proof: Assume V_1, V_2, and V_3 are the partition sets of a digraph D. Let $x, y \in V_1$, so x and y are not adjacent. By definition of D, every vertex $v \in V(D) - V_1$, $[x, v], [y, v] \in A(D)$. If there is a vertex $v \in V\ (D) - V_1$ such that $(x, v), (v, y) \in A(D)$, then (a) holds. Assuming (a) does not hold, it follows that for all $v \in$ V(D) $- V_1$ with $(x, v), (y, v) \in A(D)$. As x and y are adjacent to all the vertices in $V_2$and V_3 of D, then for any vertex $v \in V_2 \cup V_3$, either $v \in N_D^+(x) \cap N_D^+(y)$ or $v \in N_D^-(x) \cap N_D^-(y)$ or $v\ \in N_D^-(x) \cap N_D^+(y)$. Define the subsets of vertices of the partition sets $V_2$and V_3 as a following $U = \{z \in V(D) - V_1 : (x, z), (y, z) \in A(D)\}$ and $U' = \{z' \in V(D) - V_1 : (z', x), (z', y) \in A(D)\}$ and $U'' = \{z'' \in V(D) - V_1 : (z'', x), (y, z'') \in A(D)\}$. With the assumption that there is no such vertex $v \in V_2 \cup\ V_3$ such that $v \in N_D^+(x) \cap N_D^-(y)$, then $U \cap U' = \emptyset$. As D is strong, then it must have an arc zz' such that $z \in U$ and $z' \in U'$, then $xzz'y$ is an (x, y) −path P' of length 3. As (a) does not hold, then P' is a shortest (x, y) −path. ■

To prove Theorem 1.3, suppose V_1, V_2, and V_3 are the partite sets of a digraph D with $\lambda(D) \geq \alpha(D) - 1 \geq 2$. Consider $D' = (V(D'), A(D'))$ be the digraph with a set of verities $V(D') = V(D) \cup \{s\}$, where $s \notin V(D)$, and the set of arcs $A(D') = A(D) \cup \{(s, v), (v, s) : v \in V(D)\}$. Thus, D' is a strong semicomplete 4-multipartite digraph with the partition sets $V_1, \ldots, V_4$, where $V_4 = \{s\}$, and $\alpha(D') = \alpha(D)$. As $\lambda(D) \geq \alpha(D) - 1$, then $\lambda(D') \geq \lambda(D) + 1 \geq \alpha(D)$. Since $\lambda(D') \geq \alpha(D')$, then, by Theorem 1.2, D' is superelerian. Let T be a spanning closed trail of D', thus $d_T^+(v)\ =\ d_T^-(v)$ for each $v \in V(D')$; moreover, $d_T^+(s) =\ d_T^-(s) = 1$. Consider $D = D' - \{s\}$, then $T - \{s\}$ is a spanning trail of D. To complete the proof of Theorem 1.3, we must ensure the in-degree and the out-degree of the vertex s, in a closed spanning trail T of D', which must be $d_T^+(s) = d_T^-(s) = 1$. By contradiction, we may assume that $d_T^+(s) =\ d_T^-(s) = 2$, then $T - \{s\}$ contains two sub-trails spanning all the vertices of D. Let $W = T_1 \cup T_2$, where $T_1 = u_1u_2 \ldots u_m$ and $T_2 = v_1v_2 \ldots v_k$ be sub-tails of $T - \{s\}$ in a digraph D. Let us work on the ends of $T_1$and T_2, $\{u_m, v_1\}$ and $\{v_k, u_1\}$. By using the arc-strong connectivity of D, we connect the vertices $\{u_m, v_1\}$ and the vertices $\{v_k, u_1\}$. Without loss of generalities, let's connect the vertices u_m and v_1 to get the trail spanning all vertices of D. By definition of D, there are two cases to consider, either $[u_m,\ v_1] \in A(D)$ or $\{u_m,\ v_1\}$ is an independent set.

Case 1 Suppose$[u_m,\ v_1] \in A(D)$. If $(u_m,\ v_1) \in A(D)$ then $W\cup\ \{(u_m,\ v_1)\}$ is a spanning trail of D. Assume $(v_1, u_m) \in A(D)$, then by Lemma 3.1, there must exist a vertex $z \in V(D)$ such that $(u_m, z), (z, v_1) \in A(D)$, which is the shortest $(u_m,\ v_1)$-path P' in D. Thus $W \cup P'$ is a spanning trail of D.

Case 2 Suppose $\{u_m,\ v_1\}$ is an independent set. Suppose $u_m,\ v_1 \in V_1$, thus for each $z \in V(D) - V_1, [u_m, z], [v_1, z] \in A(D)$. As D is strong there must exist a vertex $z \in V(D) - V_1$, such that $(u_m, z), (z, v_1) \in A(D)$, which is the shortest $(u_m,\ v_1)$− path P'in D. Therefore $W\cup P'$ is a spanning trail of D. Assume there is no vertex satisfying $(u_m, z), (z, v_1) \in A(D)$, then by Lemma 3.2 (a) fails; so, Lemma 3.2 (b) holds. Then there exists a shortest (u_m, v_1)− path P' in D. Therefore, $W\cup P'$is a spanning trail of D. Similarly, we can join the vertices $\{v_k, u_1\}$ by the shortest (v_k, u_1) −path, say P''. Now, P' and P'' are arc-disjoint paths in D, then $W \cup P' \cup P''$ is a spanning closed trail of D. It follows that D is superelerian, under condition $\lambda(D) \geq \alpha(D) - 1$, which contracts Theorem 1.2. Thus, in a closed spanning trail T of D', $d_T^+(s) = d_T^-(s) = 1$. Therefore, $T - \{s\}$ is a spanning trail of a digraph D. This completes the proof of Theorem 1.3. ■

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory. Springer, New York, 2008.

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd Edition. Springer- Verlag, London, 2009.

[3] Y. Hong, H.-J. Lai and Q. Liu. "Supereulerian Digraphs." Discrete Math 330 (2014): 87-95.

[4] Y. Hong, Q. Liu and H.-J. Lai. "Ore-type degree condition of supereulerian digraphs." Discrete Math 339 (2016): 2042-2050.

[5] J. Bang-Jensen and A. Maddaloni, "Sufficient conditions for a digraph to be supereulerian." Graph Theory 79 (1) (2015): 8-20.

[6] J. Bang-Jensen, H. Déprés and A. Yeo. "Spanning eulerian subdigraphs avoiding k prescribed arcs in tournaments." Discrete Math 343 (2020): 112-129.

[7] J. Bang-Jensen, F. Havet and A. Yeo. "Spanning eulerian subdigraphs in semicomplete digraphs." Graph Theory 100 (2) (2022): 294-314.

[8] F. Liu, Z.-X. Tian and D. Li. "Supereulerian Locally Semicomplete Multipartite Digraphs." International J. Math. Combin 2 (2017): 123-128.

[9] V. Chvátal and P. Erdös. "A note on Hamiltonian circuits. " Discrete Math 2 (1972): 111-113.

[10] C. Thomassen. "Long cycles in digraphs." Proc. London Math. Soc 42 (1981): 231-251.