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Directed Graphs of Frobenius Companion Matrices of Order 2

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A B S T R A C T
In this work, we investigate Frobenius companion matrices with entries from the ring of integers modulo a prime number. We introduce a mapping ψ to construct a directed graph, where vertices represent these matrices and edges are defined by ψ. Our analysis focuses on the structure and properties of this directed graph, including vertex degrees, cycles, and connected components, in relation to the eigenvalues of the matrices.

Keywords: *Frobenius companion matrices, Digraphs, Eigenvalues, Characteristic polynomial, Graph connectivity*.

الرسوم البيانية الموجهة لمصفوفات المرافق لفروبنيوس ذات البعد 2

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الملخص
في هذا العمل، ندرس مصفوفات المرافق لفروبنيوس بمدخلات من حلقة الأعداد الصحيحة بمقياس عدد أولي. نقدم راسم ψ لإنشاء رسم بياني موجه، حيث تمثل الرؤوس هذه المصفوفات و تعرف الحواف بواسطة ψ. يركز تحليلنا على بنية وخصائص هذا الرسم البياني الموجه، بما في ذلك درجات الرؤوس، الحلقات والمركبات المترابطة، فيما يتعلق بالقيم الذاتية للمصفوفات.

الكلمات المفتاحية: مصفوفات فروبينيوس المرافقة، الرسوم البيانية الموجهة، القيم الذاتية، كثيرة الحدود المميزة، ترابط الرسوم البيانية.

1. Introduction

Frobenius companion matrices, named after the German mathematician Ferdinand Georg Frobenius, have a rich history rooted in the systematic study of linear algebra and polynomial equations that began in the 19th century with foundational work by mathematicians such as Arthur Cayley and James Joseph Sylvester [1-3]. Frobenius himself made significant contributions to this field, focusing on matrix theory, determinants, and group theory. One of his key contributions was the introduction of the Frobenius companion matrix, which he detailed in his paper [4] that systematically studied the characteristic polynomial of a matrix.

The Frobenius companion matrix is defined in terms of the coefficients of its associated polynomial, establishing a fundamental connection between these matrices and polynomial algebra. This connection has been further explored in the work of B. Eastman, I.-J. Kim, B.L. Shader, and K.N. Vander Meulen [5], examine the structural patterns within companion matrices and their implications for linear algebra and polynomial theory.

Using directed graphs to represent and analyze Frobenius companion matrices provides a powerful method for visualizing and understanding their properties. This approach bridges linear algebra with graph theory, offering new perspectives and tools for exploring the behavior and characteristics of systems modeled by these matrices. The study of Frobenius companion matrices, especially of type 2×2, is crucial due to their fundamental role in linking polynomial algebra and matrix theory.

Despite the well-established theoretical importance of Frobenius companion matrices, there is a compelling need to explore novel methods to visualize and analyze these matrices, enhancing our understanding and expanding their practical applications. One such novel approach is the use of directed graphs to represent Frobenius companion matrices, which can provide deeper insights into their structure and properties.

In this work, we define the set of all Frobenius companion matrices of order 2 with entries from the ring of integers modulo a prime p, denoted as $M_2^{(F)}(\mathbb{Z}_p)$. Furthermore, we introduce a mapping ψ that reflects a directed graph G_p, where the vertices $V(G_p)$ correspond to the elements of $M_2^{(F)}(\mathbb{Z}_p)$ and directed edges connect matrices in A and $\psi(A)$ for all $A \in M_2^{(F)}(\mathbb{Z}_p)$. Understanding the structure and properties of this directed graph G_p, including the degrees of vertices, cycles, and connected components, offers important insights into the characteristics of these matrices and their associated polynomials.

The ring of integers modulo a prime p, denoted as $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, is the set of integers $\{0, 1, 2, \ldots, p-1\}$ with addition and multiplication performed modulo p. This algebraic structure provides the foundation for the discussion of Frobenius companion matrices that follow.

The Frobenius companion matrix with entries from the ring $\mathbb{Z}_p$ is defined as:

$$C_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

A key property of this matrix is that the entries in C_n correspond to the coefficients of its characteristic polynomial: $p_n(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$. This critical connection between the Frobenius companion matrix and its associated polynomial is the focus of further analysis.

Consider the specific case of $n = 2$, the Frobenius companion matrix becomes:

$$C_2 = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

The characteristic polynomial of this matrix $A = C_2$ is obtained by computing the determinant of $(A - xI)$, which results in the polynomial $x^2 + ax + b$. Importantly, the eigenvalues of the matrix A are the roots of this polynomial.

Based on this foundation, we can define the set of all Frobenius companion matrices as $M_2^{(F)}(\mathbb{Z}_p) = \left\{\begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix} : a,\ b \in \mathbb{Z}_p\right\}$. Furthermore, a mapping ψ can be defined by $\psi(A) = \begin{pmatrix} 0 & 1 \\ -ab & a+b \end{pmatrix}$ that reflects a directed graph G_p, where the vertices $V(G) = M_2^{(F)}(\mathbb{Z}_p)$ and there exists a directed edge from matrix A to the matrix $\psi(A)$ for all $A \in M_2^{(F)}(\mathbb{Z}_p)$.

Understanding the structure and properties of this directed graph G_p, based on the mapping ψ and its connection to the eigenvalues of the Frobenius companion matrices, is crucial for further analysis. This concept of representing matrices using directed graphs (digraphs) provides a valuable visual and analytical tool to explore their underlying characteristics. For more information on the standard notations and principles of graph theory used here, refer to the sources [6, 7].

This work investigates the directed graph G_p defined by the mapping ψ on the set of Frobenius companion matrices with entries in the commutative ring $\mathbb{Z}_p$. The focus is on exploring the degrees of vertices, cycles, and connected components within this graph, which offers important insights into the structure and properties of these matrices and their associated polynomials. Interestingly, the work in [8] explores a similar association, examining how the structural properties of directed graphs (digraphs) can provide valuable insights into the underlying algebraic structure of commutative rings.

2. Main Results

Since G_p is a directed graph, the indegree of a vertex $A \in G_p$, is the number of directed edges coming into A, and the outdegree of A is the number of directed edges leaving the vertex A. By the definition of the mapping ψ, the outdegree of each vertex of G_p is equal one. The following proposition addresses the incoming degree of a vertex in this graph.

Proposition 1. The incoming degree of a vertex $A \in G_p$ is equal to the number of distinct roots of the characteristic polynomial of A.

Proof: Let $A = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix}$ be a vertex in the digraph G_p. Then the vertex A has an incoming edge from another vertex, let us say $B = \begin{pmatrix} 0 & 1 \\ d & c \end{pmatrix}$ if and only if the characteristic polynomial of the matrix A, written as $x^2 - (c+d)x + cd$, is always reducible modulo any prime number p. This is because the existence of an incoming edge from B to A implies that the characteristic polynomial $x^2 - ax + b$ can be factored as a product of two lower-degree polynomials modulo p, Therefore, the incoming degree of A is equal to the number of elements c in $\mathbb{Z}_p$ such that the characteristic polynomial of A is reducible modulo p.□

This proposition provides a precise characterization of the incoming degree of vertices in the digraph G_p in terms of the reducibility of their associated characteristic polynomials. We observe that the eigenvalues of the matrix $A \in G_p$ are determined by its characteristic polynomial $\lambda^2 - a\lambda + b$. Since $\mathbb{Z}_p[\lambda]$ is a unique factorization domain, the number of distinct eigenvalues of this matrix can be 0, 1, or 2, depending on the values of a and b modulo p:

- If the discriminant a^2-4b is not a quadratic residue modulo p, then the characteristic polynomial $p_A(x)$ has no roots in $\mathbb{Z}_p$, and A has no eigenvalues. Therefore, the incoming degree of A is 0.
- If the discriminant a^2-4b is 0 modulo p, then the characteristic polynomial $p_A(x)$ has a single repeated root in $\mathbb{Z}_p$, and the eigenvalues of A are equal. Therefore, the incoming degree of A is 1.
- If the discriminant a^2-4b is a quadratic residue modulo p, then the characteristic polynomial $p_A(x)$ has two distinct roots in $\mathbb{Z}_p$, and the eigenvalues of A are distinct. Therefore, the incoming degree of the vertex A in the digraph G_p is 2.

The key relationship is that the eigenvalues of the matrix $\psi(A)$ are exactly the entries a and $-b$ from the original matrix A. Additionally, another matrix $\begin{pmatrix}0 & 1\\ a & -b\end{pmatrix}$ is also pointed to $\psi(A)$ by the function ψ, where $\psi(\begin{pmatrix}0 & 1\\ a & -b\end{pmatrix})=\psi(A)$. This means the incoming degree of A is determined by the factorization of its characteristic polynomial over the finite field $\mathbb{Z}_p$, and the construction of $\psi(A)$ directly utilizes the coefficients a and b from the original matrix A.
We conclude that the defined map ψ simply transforms a matrix A by computing the product of the coefficients and the sum of the coefficients. This does not change the number of eigenvalues, and hence the indegree remains 0, 1, or 2.

Definition. The sequence

$$\begin{pmatrix}0 & 1\\ -b_1 & a_1\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -b_2 & a_2\end{pmatrix}\to\cdots\to\begin{pmatrix}0 & 1\\ -b_n & a_n\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -b_1 & a_1\end{pmatrix}$$

of arrows in G_p defines a cycle of length n ($n-cycle$) if for all $1\le i<n$, we have

$$\begin{pmatrix}0 & 1\\ -b_i & a_i\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -a_ib_i & a_i+b_i\end{pmatrix}.$$

A cycle of length one is referred to as a loop, and we call any loop with indegree and outdegree equal to one is an isolated loop. Since the ring $\mathbb{Z}_p$ doesn't contain zero divisors, thus the only isolated loop in G_p for any prime number is $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.

Different types of loop structures, including loops(isolated), and k-cycles are presented in the digraph G_5 in Figure 1.

Regarding the definition of loops in the graph G_p, we have the following proposition:

Proposition 2. The number of loops in the graph G_p corresponds to singular matrices, that is exactly $n=\#(\mathbb{Z}_p)=p$.

Proof: Each element a in the set $\mathbb{Z}_p$ corresponds to a singular matrix $A=\begin{pmatrix}0 & 1\\ 0 & a\end{pmatrix}$ in the graph G_p. By the definition of the map ψ, we have $A\circlearrowleft$, which is a self-loop. Since there is a one-to-one correspondence between the elements of $\mathbb{Z}_p$ and the singular matrices in G_p. Therefore, the graph G_p contains exactly $n=\#(\mathbb{Z}_p)=p$ loops that correspond to the vertices $\begin{pmatrix}0 & 1\\ 0 & a\end{pmatrix}$. □

Regarding the vertices in $V(G_p)$ with no incoming edges, we observe that these vertices are precisely the ones with irreducible characteristic polynomials.
From the definition of ψ, if a vertex v in a connected subgraph of G_p with an outgoing degree of one belongs to two different cycles, then these cycles must be the same cycle because v can only point to one other vertex. This implies that all vertices that follow from v are part of both cycles, making them identical. In the context of graph theory, a connected graph is one in which there is a path between any two distinct vertices. Consequently, a connected component of this graph, denoted C_p, is a maximal connected subgraph. This observation regarding the decomposition of a graph's vertex set into disjoint connected components leads to a general property of the directed graph representing any mapping ψ.

Proposition 3. Each component has exactly one cycle, and the number of components in G_p is equal to the number of its cycles.

Proof: From the definitions of the map ψ, we observe that every component must terminate in a cycle. Since there is no path between any two cycles in a connected component and cycles cannot share vertices, each component thus contains only one cycle. Therefore, the number of components in G_p is equal to the number of its cycles. □

It should be noted that every vertex $A=\begin{pmatrix}0 & 1\\ -b & a\end{pmatrix}$ in the digraph G_p has a characteristic polynomial x^2-ax+b with coefficients $a,b\in\mathbb{Z}_p$. The roots of this polynomial represent the eigenvalues of A. According to the previous discussion, the vertices that point to (coming into) A are $\begin{pmatrix}0 & 1\\ \lambda_2 & \lambda_1\end{pmatrix}$, and $\begin{pmatrix}0 & 1\\ \lambda_1 & \lambda_2\end{pmatrix}$. Therefore, the reducibility of the characteristic polynomial of any matrix $A\in V(G_p)$ is crucial for determining the vertices that point to A. To examine some properties of k-cycles in this digraph more closely, we introduce the following proposition, with a proof similar to that of Proposition 3.2 in [9].

Proposition 4. Let $\begin{pmatrix}0 & 1\\ -b_1 & a_1\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -b_2 & a_2\end{pmatrix}\to\cdots\to\begin{pmatrix}0 & 1\\ -b_n & a_n\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -b_1 & a_1\end{pmatrix}$ is a cycle of length n in G_p, then,

1. $\sum_{i=1}^{n} b_i = 0$.
2. $\prod_{i=1}^{n} a_i = 1$.

Proof: Based on the relation between the matrices and their characteristic polynomials, there is an arrow $\begin{pmatrix} 0 & 1 \\ -b_{i-1} & a_{i-1} \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -b_i & a_i \end{pmatrix}$ if and only if one has the equality $x^2 - a_i x + b_i = (x - a_{i-1})(x - b_{i-1})$ in the polynomial ring $\mathbb{Z}_p[x]$. Therefore, the cycle condition implies the equality:

$$\prod_{i=1}^{n} x^2 - a_i x + b_i = \prod_{i=1}^{n} (x - a_{i-1})(x - b_{i-1})$$

in the polynomial ring $\mathbb{Z}_p[x]$. Performing a straightforward multiplication, the result is as follows:

$$\prod_{i=1}^{n} (x^2 + a_i x + b_i) = x^{2n} - [a_1 + a_2 + \cdots + a_n]x^{2n-1} + [(a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n) + (b_1 + b_2 + \cdots + b_n)]x^{2n-2} + \cdots + (b_1 b_2 \ldots b_{n-1} b_n)$$

$$\prod_{i=1}^{n} (x - a_{i-1})(x - b_{i-1}) = x^{2n} - [(a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)]x^{2n-1} + [(a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n) + (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) + (b_1 b_2 + b_1 b_3 \ldots + b_{n-1} b_n)]x^{2n-2} + \cdots + (a_1 a_2 \ldots a_n)(b_1 b_2 \ldots b_n)$$

By comparing the coefficients, we can initially obtain $\sum_{i=1}^{n} b_i = 0$. The constant term on both sides yields $(b_1 b_2 \ldots b_{n-1} b_n) = (a_1 a_2 \ldots a_{n-1} a_n)(b_1 b_2 \ldots b_{n-1} b_n)$.

That implies $\prod_{i=1}^{n} a_i = 1$.☐

Remarks:

i. Since any composite number n can be factored into a product of prime factors, the analysis of the directed graph G_p can be expanded to the set $M_2^{(F)}(\mathbb{Z}_n)$ (the set of Frobenius companion matrices with entries in the integers modulo n). This generalization to the integers modulo n enables a deeper analysis of the structural properties and interrelationships within this expanded set $M_2^{(F)}(\mathbb{Z}_n)$.

ii. For any matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is presented as $p_B(x) = x^2 - Trace(B)x + Det(B)$, So the corresponding Frobenius companion matrix of B can be written in the form $\begin{pmatrix} 0 & 1 \\ -Det(B) & Trace(B) \end{pmatrix}$. This observation allows us to define a digraph associated with the non-commutative ring of matrices with entries in the ring of integers modulo n.

3. Conclusion

The study defines Frobenius companion matrices of size 2×2 with entries from the ring of integers modulo a prime p and introduces a mapping ψ to represent them as a directed graph G_p. The focus is on analyzing the structure and properties of G_p, including vertex degrees, cycles, and connected components, to gain insights into the matrices and their associated polynomials. Key findings include the relationship between vertex indegree and the roots of characteristic polynomials, as well as the correspondence between vertices and eigenvalues. Additionally, the study identifies the number of loops in G_p and highlights the significance of irreducible characteristic polynomials for vertices without incoming edges. The study concludes by establishing a connection between components and cycles in G_p, providing a comprehensive understanding of the graph's structure and its implications for matrix analysis.

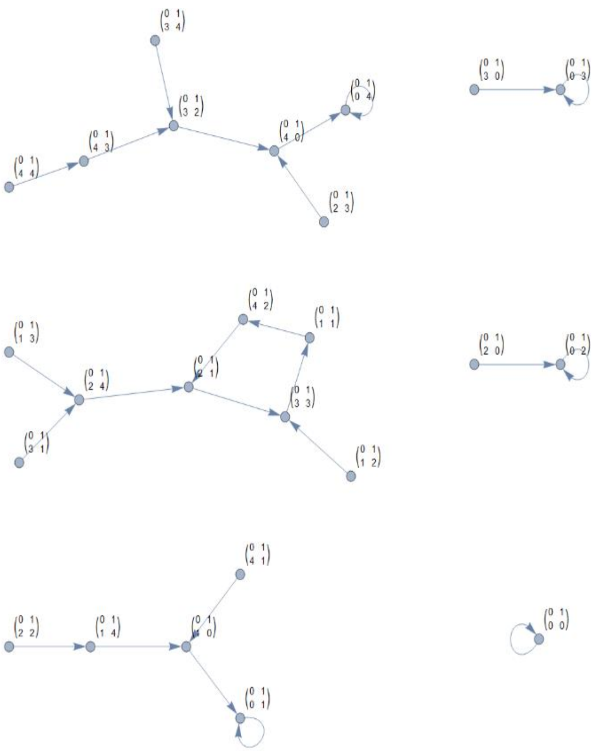

Figure 1 The digraph of G_5

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