

Product Embedding Methods for Free (Para)Topological Groups

Ali S Elfard ^a 

^aDepartment of Mathematics, Faculty of Sciences, University of Zawia, Zawia, Libya

*Corresponding email address: aelfard@zu.edu.ly

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ABSTRACT

Product embedding methods constitute one of the fundamental techniques in the construction of free topological groups and in the study of their universal properties. In this paper, we investigate the role of these methods in the construction of free topological and free paratopological groups. Beginning with the classical constructions of Kakutani and Hewitt–Ross, we analyze the use of separating families of continuous mappings, evaluation maps, diagonal embeddings, and product representations as the main ingredients in the existence and structural characterization of free objects. We then examine Elfard’s adaptation of this approach to free paratopological groups, where the absence of continuous inversion requires substantial modifications of the classical arguments while preserving the universal mapping property. A comparison between the topological and paratopological constructions is given, highlighting both the common embedding framework and the essential differences arising from the weaker algebraic-topological assumptions. The analysis demonstrates that product embeddings provide a unifying mechanism for free (para)topological group constructions and suggests possible extensions of these methods to categories with weaker continuity conditions, such as quasitopological and semitopological groups.

Keywords: Free Topological Group, Free Paratopological Group, Product Embedding, Evaluation Map, Separating Family, Universal Mapping Property.

طرق الدفن الجدائي للزمر (شبه) التوبولوجية الحرة

علي السيد الفرد^a

^aقسم الرياضيات، كلية العلوم، جامعة الزاوية، الزاوية، ليبيا

الملخص

تشكل طرق الدفن الجدائي إحدى التقنيات الأساسية في بناء الزمر التوبولوجية الحرة وفي دراسة خواصها الشاملة. نبحث في هذه الورقة دور هذه الطرق في بناء الزمر التوبولوجية الحرة والزمر شبه التوبولوجية الحرة. وبدءاً من التركيبات الكلاسيكية لكاكوتاني و هيويت-روس، نحلل استخدام عائلات فاصلة من الدوال المستمرة، ودوال التقييم، والدفن القطري، وتمثيلات الجداء بوصفها المكونات الرئيسية في وجود الأجسام الحرة وتوصيفها البنيوي. ثم نخصص تكييف Elfard لهذا المنهج للزمر شبه التوبولوجية الحرة، حيث يتطلب غياب تطبيق المعكوس المستمر تعديلات جوهرية في الحجج الكلاسيكية مع الحفاظ على خاصية التعيين الشاملة. ونُقارِم مقارنة بين التركيبات التوبولوجية والتركيبات شبه التوبولوجية، مع إبراز كل من إطار الدفن المشترك والاختلافات الجوهرية الناشئة عن الافتراضات الجبرية-التوبولوجية الأضعف. وبيّن التحليل أن الدفن الجدائي يوفر آلية موحدة لبناء الزمر (شبه) التوبولوجية الحرة، وتقترح امتدادات ممكنة لهذه الطرق إلى مجموعات ذات شروط استمرارية أضعف، مثل الزمر الكوازي توبولوجية والنصف توبولوجية.

الكلمات المفتاحية: زمرة توبولوجية حرة، زمرة شبه توبولوجية حرة، دفن جدائي، دالة تقييم، عائلة فاصلة، خاصية الدالة الشاملة.

1. Introduction

The theory of free topological groups is one of the fundamental developments in topological algebra, providing a natural extension of the classical free group construction to categories where algebraic and topological structures interact. Given a topological space X , the problem is to construct a group containing X as a subspace and satisfying a universal mapping property with respect to continuous mappings into topological groups. This problem was initiated by

Markov [1], who introduced free topological groups over Tychonoff spaces. A central role in the existence and structural analysis of these groups is played by product embedding techniques. In particular, the classical constructions of Kakutani [2] and Hewitt–Ross [3] demonstrated that separating families of continuous mappings, evaluation maps, and diagonal embeddings provide powerful tools for representing spaces inside suitable products of topological groups

and for establishing the universal properties of free objects.

A natural generalization of topological groups is given by paratopological groups, where only the continuity of multiplication is required, and the inversion operation is not assumed to be continuous. Although every topological group is a paratopological group, the lack of continuous inversion creates essential differences and prevents a direct application of many classical arguments. The existence of free paratopological groups was established through the work of Romaguera, Sanchis, and Tkachenko [4], and further developed by Ravsky [5], [6], Pyrch and Ravsky [7], Elfard and Nickolas [8], [9], and Elfard [10], [11], [12]. In particular, Elfard [11] adapted the product embedding method of Kakutani and Hewitt–Ross to the paratopological setting. This adaptation requires new separation arguments and suitable families of mappings into paratopological groups, since the continuity of inversion, which is available in the classical case, can no longer be used.

The purpose of this paper is to study the product embedding methods underlying the constructions of free topological and free paratopological groups from a common perspective. We analyze the role of separating families, evaluation mappings, and product representations in the constructions of Kakutani, Hewitt–Ross, and Elfard, emphasizing both their shared framework and the essential modifications required in the absence of continuous inversion. After recalling the basic notions and embedding principles, we review the classical constructions for free topological groups and then examine Elfard’s construction of free paratopological groups in detail. The comparison presented in this paper illustrates that product embeddings form a unifying method for constructing free objects in different categories of topological algebra and suggests possible extensions of these techniques to weaker structures, such as quasitopological and semitopological groups.

2. Preliminaries

We begin by recalling some basic notions and results concerning paratopological groups and topological groups that will be used throughout the paper. In particular, we emphasize the role of translations and neighborhood systems, which are essential in the study of the topology of free paratopological groups.

Definition 2.1. A paratopological group is a group G equipped with a topology such that the multiplication mapping $m: G \times G \rightarrow G, (x, y) \mapsto xy$, is continuous. If, in addition, the inversion mapping $i: G \rightarrow G, x \mapsto$

x^{-1} , is continuous, then G is called a topological group.

Every topological group is therefore a paratopological group, while the converse does not hold in general. A standard example is the additive group $(\mathbb{R}, +)$ endowed with the lower limit topology (the Sorgenfrey topology). The addition operation is continuous, so it is a paratopological group. However, the inversion mapping $x \mapsto -x$ is not continuous, since the inverse image of the basic neighborhood $[0, 1)$ of 0 is $(-1, 0]$, which is not open in the Sorgenfrey topology. Hence this example is not a topological group.

Several results describe conditions under which a paratopological group is necessarily a topological group. Ellis [13] proved that every locally compact Hausdorff paratopological group is a topological group. Ravsky [5] showed that every pseudocompact Hausdorff paratopological group is a topological group. Moreover, Arhangel’skii and Tkachenko [14] proved that every regular Baire paratopological group is a topological group.

Translations are among the basic tools in the study of (para)topological groups, since they connect the algebraic structure with the topology.

Definition 2.2. Let G be a paratopological group and let $a \in G$. The left translation by a and the right translation by a are defined, respectively, by $L_a: G \rightarrow G, L_a(x) = ax$, and $R_a: G \rightarrow G, R_a(x) = xa$.

Proposition 2.3. Let G be a paratopological group and let $a \in G$. Then the maps L_a and R_a are homeomorphisms of G onto itself.

Proof. The continuity of L_a and R_a follows immediately from the continuity of multiplication. Moreover, $L_a^{-1} = L_{a^{-1}}, R_a^{-1} = R_{a^{-1}}$, and these maps are also continuous. Hence both translations are homeomorphisms. $\square$

Definition 2.4. A topological space X is called homogeneous if, for every pair of points $x, y \in X$, there exists a homeomorphism $h: X \rightarrow X$ such that $h(x) = y$.

Remark 2.5. Every paratopological group is homogeneous. Indeed, if $x, y \in G$, then the left translation $L_{yx^{-1}}: G \rightarrow G$ is a homeomorphism satisfying $L_{yx^{-1}}(x) = yx^{-1}x = y$. Therefore, the topology of a paratopological group can be studied by considering the neighborhood system at the identity element.

The following characterization of neighborhood bases

at the identity is a standard result in the theory of paratopological groups; see [5] and [14].

Theorem 2.6 (Neighborhood system in a paratopological group). *Let G be a paratopological group with identity element e , and let $\mathcal{H}$ be a family of neighborhoods of e . Then $\mathcal{H}$ is a neighborhood base at e if and only if the following conditions hold:*

1. *For every $U, V \in \mathcal{H}$, there exists $W \in \mathcal{H}$ such that $W \subseteq U \cap V$.*
2. *For every $U \in \mathcal{H}$, there exists $V \in \mathcal{H}$ such that $VV \subseteq U$.*
3. *For every $U \in \mathcal{H}$ and every $x \in U$, there exists $V \in \mathcal{H}$ such that $xV \subseteq U$.*
4. *For every $U \in \mathcal{H}$ and every $g \in G$, there exists $V \in \mathcal{H}$ such that $gVg^{-1} \subseteq U$.*

Remark 2.7. *For a topological group, the preceding conditions must be supplemented by the additional requirement that for every $U \in \mathcal{H}$ there exists $V \in \mathcal{H}$ such that $V^{-1} \subseteq U$. This additional condition reflects the continuity of the inversion mapping, which is not required in a paratopological group.*

By homogeneity, a neighborhood base at the identity determines the topology of the whole group. More precisely, if $\mathcal{H}$ is a neighborhood base at the identity e of a paratopological group G , then $\{xU : x \in G, U \in \mathcal{H}\}$ is a neighborhood base at each point $x \in G$. Consequently, a subset $O \subseteq G$ is open if and only if for every $x \in O$ there exists $U \in \mathcal{H}$ such that $xU \subseteq O$.

Definition 2.8. *Let X be a nonempty set. The abstract free group on X , denoted by $F_a(X)$, is the group generated by the elements of X subject to no relations other than those required by the group axioms. Equivalently, the abstract free group on X , denoted by $F_a(X)$, is a group together with an injective mapping $i: X \rightarrow F_a(X)$ satisfying the following universal property: for every group G and every mapping $f: X \rightarrow G$, there exists a unique group homomorphism $\hat{f}: F_a(X) \rightarrow G$ such that $\hat{f} \circ i = f$.*

$$\begin{array}{ccc} X & \xrightarrow{i} & F_a(X) \\ & \searrow f & \downarrow \hat{f} \\ & & G \end{array}$$

The elements of $F_a(X)$ are called *reduced words*. To describe them explicitly, let $X^{-1} = \{x^{-1} : x \in X\}$ be a disjoint copy of X , whose elements are regarded as formal inverses of the elements of X . The set $\tilde{X} = X \cup X^{-1} \cup \{e\}$, where $e \notin X$ denotes the identity element. A word over X is a finite sequence $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$, where $x_i \in X$ and $\varepsilon_i \in \{1, -1\}$ for each i . The word is

said to be *reduced* if it contains no adjacent pair of the form xx^{-1} or $x^{-1}x$, for any $x \in X$. Every word can be transformed into a unique reduced word by successively deleting adjacent inverse pairs. The group operation in $F_a(X)$ is defined by concatenating two reduced words and then reducing the resulting word. The identity element is the empty word, and the inverse of a reduced word $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ is $w^{-1} = x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \dots x_1^{-\varepsilon_1}$, followed by reduction if necessary. The number of letters in a reduced word w is called its *length* and is denoted by $l(w)$. By convention, $l(e) = 0$.

The abstract free group $F_a(X)$ captures the algebraic structure generated by X but does not reflect its topology. When X is a topological space, it is natural to seek a topology on $F_a(X)$ that is compatible with the group operations and extends the topology of X . This leads to the notions of free topological and free paratopological groups, which equip $F_a(X)$ with topologies satisfying the appropriate universal mapping properties.

3. Free Topologies on Abstract Free Groups

The concept of free topological groups originates from the classical notion of free groups in algebra, where a set X generates an abstract free group satisfying the corresponding universal mapping property. Motivated by incorporating topological structure into this construction, Markov [1] introduced free topological groups and proved that every Tychonoff space X admits a free topological group $F(X)$. As stated in Definition 3.1, the space X is embedded in $F(X)$, generates $F(X)$ algebraically, and every continuous mapping from X into a topological group extends uniquely to a continuous homomorphism from $F(X)$. This universal property makes $F(X)$ the canonical free object in the category of topological groups and provides a powerful framework for relating the topology of X to the algebraic structure of the generated group. Since Markov's pioneering work, the theory has been developed extensively. Graev extended the construction to pointed spaces and introduced the Graev extension of continuous pseudometrics, while Kakutani and Hewitt–Ross developed product embedding methods that have become fundamental in the study of free topological groups. Further contributions by Joiner, Arhangel'skiĭ, Tkachenko, Nickolas, Yamada, and others established important results on neighborhood bases, embedding properties, finite-level subspaces, metrizability, and related structural aspects.

Definition 3.1. Let X be a Tychonoff space. The free topological group $F(X)$ over X is a topological group endowed with a continuous embedding of X into $F(X)$ such that the following conditions are satisfied:

1. X is a subspace of $F(X)$ under the induced topology;
2. the set X algebraically generates $F(X)$ as a group;
3. for every continuous mapping $f: X \rightarrow G$ into an arbitrary topological group G , there exists a unique continuous homomorphism $\hat{f}: F(X) \rightarrow G$ extending f .

Motivated by Markov's construction of free topological groups [1], Romaguera, Sanchis, and Tkachenko [4] introduced free paratopological groups and established their existence over arbitrary topological spaces. Subsequent work by Ravsky [5], [6] and Pyrch and Ravsky [7] further contributed to the development of the theory. The theory was further advanced through a series of papers by Elfard [10], [11], [12] and by Elfard and Nickolas [8], [9], which established several fundamental results concerning free paratopological groups. We conclude this section by recalling the definition of a free paratopological group.

Definition 3.2. Let X be a topological space. The free paratopological group $FP(X)$ over X is a paratopological group endowed with a continuous embedding of X into $FP(X)$ such that the following conditions are satisfied:

1. X is a subspace of $FP(X)$ under the induced topology;
2. the set X algebraically generates $FP(X)$ as a group;
3. for every continuous mapping $f: X \rightarrow G$ into an arbitrary paratopological group G , there exists a unique continuous homomorphism $\hat{f}: FP(X) \rightarrow G$ extending f .

4. Embedding Methods via Separating Families

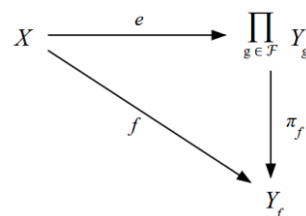
Let X be a topological space, and let $\mathcal{F} = \{f: X \rightarrow Y_f\}$ be a family of continuous mappings, where each Y_f is a topological space depending on f . The family $\mathcal{F}$ is said to *separate* (or *distinguish*) *points* if, for every pair of distinct points $x, y \in X$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. It is said to *separate points and closed sets* if, whenever A is a closed subset of X and $x \in X \setminus A$, there exists $f \in \mathcal{F}$ such that

$$f(x) \notin cl_{Y_f}(f(A)).$$

Associated with $\mathcal{F}$ is the *evaluation* (or *diagonal*) map

$$e: X \rightarrow \prod_{f \in \mathcal{F}} Y_f, \quad e(x) = (f(x))_{f \in \mathcal{F}},$$

which satisfies $\pi_f \circ e = f$ for every $f \in \mathcal{F}$, where π_f denotes the canonical projection onto the f -th coordinate.



Thus, the evaluation map simultaneously records the values of all functions in $\mathcal{F}$ and provides a canonical representation of X in the product space. In particular, if $\mathcal{F}$ separates points, then e is injective, while if $\mathcal{F}$ separates points and closed sets, then e is a topological embedding of X into the product space $\prod_{f \in \mathcal{F}} Y_f$.

The evaluation map is a fundamental tool in the product embedding methods used throughout this paper. By associating to each point of a topological space the tuple of its images under a family of continuous mappings, it provides a canonical embedding into a product space whenever the family separates points and closed sets. This characterization, established by Kelley's Embedding Lemma in [15], forms one of the basic techniques for representing topological spaces as subspaces of products and has numerous applications in general topology. In the setting of free (para)topological groups, these embedding methods play a central role in the construction of free objects and the verification of their universal properties. The proof of the lemma is omitted, as it follows the argument of Kelley [15].

Lemma 4.1. (Kelley's Embedding Lemma). Let $\mathcal{F}$ be a family of continuous functions, where each $f \in \mathcal{F}$ maps a topological space X into a topological space Y_f . Then:

1. The evaluation map $e: X \rightarrow \prod_{f \in \mathcal{F}} Y_f$ is continuous.
2. If $\mathcal{F}$ distinguishes points and closed sets, then e is an open mapping of X onto its image $e[X]$.
3. The map e is one-to-one if and only if $\mathcal{F}$ distinguishes points.

5. Construction of Free Topological Groups via Product Embeddings

The embedding principle established in the previous section forms the foundation of the classical product constructions of free topological groups. By embedding the generating space into a suitable product of topological groups through evaluation maps, Kakutani obtained the first general existence theorem

for free topological groups. Hewitt and Ross later refined this approach, replacing the original construction with a more flexible product embedding technique that has become the standard method in the theory.

5.1 Kakutani's Product Embedding Construction

A topological group G is *toroidal* if it is topologically isomorphic to the infinite direct product topological group $\prod_{\gamma \in \Gamma} G_\gamma$, where each G_γ is a topological group topologically isomorphic to the additive group K of all real numbers mod 1 with the usual compact topology. The following embedding lemmas, proved by Kakutani [2], are fundamental to the modern theory of free topological groups. They establish embeddings of completely regular spaces into toroidal and totally bounded topological groups, where, in the terminology of Kakutani, a toroidal group is a topological group that is topologically isomorphic to a subgroup of a product of circle groups. These embedding results reduce the study of arbitrary completely regular spaces to well-structured topological groups and form the foundation of Kakutani's direct proof of the existence of free topological groups. They have since become a fundamental tool in embedding theory and in the universal construction of topological groups.

Lemma 5.1.1. *Every completely regular topological space is a subspace of a toroidal group.*

Lemma 5.1.2. *Every completely regular topological space X is a subspace of a totally bounded topological group G such that $\psi(G) \leq \max(\psi(X), \aleph_0)$.*

The following Theorem 5.1.3, due to Kakutani [2], is a fundamental existence result in the theory of free topological groups. It establishes the existence of a free topological group over every completely regular space via a direct construction and characterizes it by its universal mapping property. This property underlies many subsequent developments and generalizations, including free abelian and free paratopological groups. We now state the theorem and briefly outline Kakutani's proof.

Theorem 5.1.3. *For any completely regular topological space X , there exists a topological group $F(X)$ with the following properties:*

1. X is a subspace of $F(X)$;
2. X generates $F(X)$ algebraically;
3. for any continuous mapping $\varphi: X \rightarrow H$ of X into any topological group H , there exists a continuous homomorphism $\Phi: F(X) \rightarrow H$ such that $\Phi(x) = \varphi(x)$, $x \in X$.

Outline of Kakutani's proof: Kakutani's proof consists of four main steps. First, a suitable product topological group is constructed from all continuous mappings of X into topological groups of bounded weight. Next, using the embedding established in Lemma 5.1.2, the space X is identified with a subspace of this product. The free topological group $F(X)$ is then defined as the topological subgroup generated by the embedded copy of X , ensuring that X generates $F(X)$ algebraically. Finally, the universal mapping property is established by showing that every continuous mapping from X into a topological group is realized as a coordinate projection of the product and therefore extends uniquely to a continuous homomorphism from $F(X)$. The uniqueness of $F(X)$ follows immediately from this universal property.

5.2 The Hewitt-Ross Product Embedding Construction

The following theorem of Hewitt and Ross [3] provides another fundamental construction of free topological groups and strengthens the existence of Kakutani. While Kakutani's theorem establishes the existence of a free topological group over every completely regular space by embedding the space into a product of topological groups and obtaining the universal mapping property, the Hewitt–Ross theorem adds the important property that the original space X is a closed subspace of the resulting free group. The proof uses a similar product construction, but it is based on a carefully selected family of representatives of topological groups with bounded weight, together with all continuous mappings from X into these groups. The additional closedness property makes the Hewitt–Ross construction particularly useful in studying embeddings and separation properties of subspaces of free topological groups. Thus, Kakutani's theorem provides the fundamental existence and universal characterization of free topological groups, whereas the Hewitt–Ross refinement gives stronger control over the topology of the embedded space and has become an essential tool in later investigations of the structure and properties of free topological groups.

We next present the Hewitt–Ross theorem, which constructs free topological groups using product embedding methods and establishes their universal property. Since the proof follows the original argument of Hewitt and Ross [3], it is omitted. Instead, we provide a brief outline of the principal steps and the underlying ideas of the construction.

Table 1: Comparison of the principal product embedding constructions

Construction	Category	Method	Contribution
Kakutani	Topological groups	Product embeddings	Existence of free topological groups.
Hewitt–Ross	Topological groups	Refined product embeddings	Universal property with stronger embedding properties.
Elfard	Paratopological groups	Adapted product embeddings	Extension to free paratopological groups without continuous inversion.

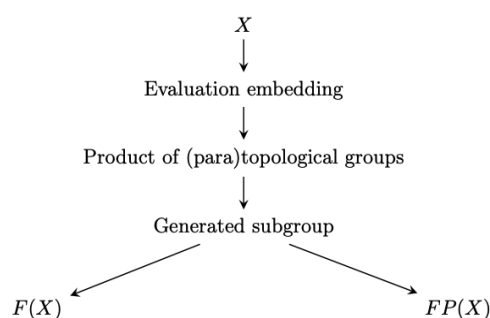
Theorem 5.2.1. *For every completely regular space X , there exists a topological group $F(X)$ with the following properties:*

1. X is a closed subspace of $F(X)$;
2. algebraically, $F(X)$ is the free group generated by X ;
3. for any continuous mapping φ of X into any topological group G , there exists a continuous homomorphism Φ of $F(X)$ into G such that $\Phi(x) = \varphi(x)$ for all $x \in X$.

Outline of the Hewitt–Ross proof: The proof follows the product embedding method introduced by Kakutani, together with additional arguments ensuring that the resulting free topological group contains X as a closed subspace. First, a representative family of topological groups of bounded weight and all continuous mappings from X into these groups are considered. These mappings determine a diagonal embedding of X into a suitable product of topological groups, allowing X to be identified with a subspace of the product. The free topological group $F(X)$ is then defined as the subgroup generated by the embedded copy of X .

To establish the algebraic freeness of $F(X)$, the proof employs finite-dimensional unitary groups and permutation matrices to separate reduced words in the free group, showing that no nontrivial reduced word represents the identity. The closedness of X in $F(X)$ is proved by constructing neighborhoods in the product group that separate points of X from elements of $F(X) \setminus X$. Finally, the universal mapping property is obtained by observing that every continuous mapping from X into a topological group appears as one of the coordinate mappings of the product and therefore extends uniquely to a continuous homomorphism from $F(X)$. The uniqueness of $F(X)$ follows immediately from this universal property.

Remark 5.2.2. *The proof of Hewitt and Ross extends Kakutani’s product embedding construction by establishing additional structural properties of the free topological group. While both proofs employ the same embedding technique and derive the universal*

**Diagram 1:** The product imbedding methods

mapping property from coordinate projections, Hewitt and Ross further prove that X is a closed subspace of $F(X)$ and that $F(X)$ is algebraically the free group generated by X , using additional arguments involving unitary groups, permutation matrices, and topological separation.

6. Construction of Free Paratopological Groups via Product Embeddings

Theorem 6.1, due to Elfard [11], establishes the existence of free paratopological groups over arbitrary topological spaces. Its proof adapts the product embedding methods of Kakutani and Hewitt–Ross for free topological groups to the category of paratopological groups. Since inversion need not be continuous in a paratopological group, the classical argument cannot be applied directly, and the construction requires a different approach while preserving the universal mapping property through suitable product embeddings and separating families of continuous mappings. Kelley’s Embedding Lemma plays a central role in embedding the underlying space into an appropriate product, from which the free paratopological group is constructed. As this theorem forms the foundation for many subsequent developments in the theory, we include the complete proof.

Before presenting the construction, [Table 1](#) and [Diagram 1](#) summarize the product embedding method underlying both the classical constructions of free topological groups and its adaptation to free paratopological groups. In each case, the generating

Table 2: Comparison of the Hewitt–Ross and Elfard constructions.

Aspect	Hewitt–Ross	Elfard
Category	Topological groups.	Paratopological groups.
Construction	Embeds X into a product of topological groups.	Embeds X into a product of paratopological groups.
Embedding	Uses a diagonal embedding and Kelley’s embedding lemma.	Uses the same embedding method.
Free group	Defines $F(X)$ as the subgroup generated by X .	Defines $FP(X)$ as the subgroup generated by X .
Algebraic freeness	Uses unitary groups and permutation matrices.	Uses mappings into an abstract free group with the indiscrete topology.
Universal property	Obtained by coordinate projections.	Obtained by the same projection argument.
Main contribution	Constructs free topological groups.	Extends the construction to paratopological groups without assuming continuous inversion.

space is embedded into a suitable product via an evaluation map, and the subgroup generated by its image satisfies the required universal property.

Let $\{\{k, k + 1, \dots\} : k \in \mathbb{Z}\}$ be a collection of subsets of the additive group $\mathbb{Z}$. Then it is a base for a paratopological group topology on $\mathbb{Z}$, and we denote the corresponding paratopological group by $\mathbb{Z}^*$.

Theorem 6.1. *Let X be a topological space. Then there exists a paratopological group $FP'(X)$ such that*

1. X is a subspace of $FP'(X)$;
2. algebraically, $FP'(X)$ is the free group generated by X ; and
3. for every continuous mapping ϕ of X into any paratopological group G , there exists a continuous homomorphism $\hat{\phi}: FP'(X) \rightarrow G$ extending ϕ .

Outline of the proof: the proof, due to Elfard, adapts the product embedding method of Kakutani and Hewitt–Ross to the category of paratopological groups. It consists of the following main steps. First, suitable families of continuous mappings from X are constructed to separate points, separate points from closed sets, and distinguish reduced words in the free group. These mappings take values in carefully chosen paratopological groups, including $\mathbb{Z}_2$, $\mathbb{Z}^*$, and the abstract free group equipped with the indiscrete topology. Next, a representative family of paratopological groups is formed, and the associated continuous mappings are combined to obtain a diagonal embedding of X into a product of paratopological groups. By Kelley’s embedding lemma, this embedding is a homeomorphism onto its image, allowing X to be identified with a subspace of the product.

The free paratopological group $FP'(X)$ is then defined as the subgroup generated by the embedded copy of X .

The algebraic freeness is established by showing that every nontrivial reduced word can be separated by a suitable homomorphism into an abstract free group. Finally, the universal property is from X into a paratopological group appears, up to topological isomorphism, as one of the coordinate mappings of the product and therefore extends uniquely to a continuous homomorphism from $FP'(X)$. The uniqueness of the free paratopological group then identifies $FP'(X)$ with $FP(X)$.

Remark 6.2. *Since Elfard’s construction is an adaptation of the Hewitt–Ross construction to the category of paratopological groups, it is instructive to compare the two proofs. Although both constructions share the same general product embedding framework, Elfard’s proof incorporates several modifications to accommodate the absence of continuity of the inversion map. A comparison highlighting the common features and the principal differences between the Hewitt–Ross and Elfard constructions is presented in Table 2.*

7. Unified Framework for Free (Para)Topological groups

The constructions of free topological groups due to Kakutani and Hewitt–Ross, and the construction of free paratopological groups due to Elfard, although developed in different settings, are based on a common embedding principle. The essential idea is to represent a topological space X inside a suitable product of groups by means of a family of continuous mappings and then define the desired free object as the subgroup generated by the embedded copy of X .

Let $\mathcal{F} = \{f_\alpha: X \rightarrow G_\alpha : \alpha \in A\}$ be a family of continuous mappings, where each G_α is a topological group or a paratopological group according to the

category under consideration. The associated evaluation mapping

$$e: X \rightarrow \prod_{\alpha \in A} G_{\alpha}, \quad e(x) = (f_{\alpha}(x))_{\alpha \in A},$$

provides a canonical representation of X in a product space. When the family $\mathcal{F}$ separates points and closed sets, Kelley's embedding lemma ensures that e is a topological embedding. The subgroup generated by $e(X)$ then becomes the natural candidate for the free object, and the universal mapping property follows from the coordinate projections of the product.

Although the underlying construction is similar, the arguments used in the three cases are not identical. Kakutani's method relies on embeddings into toroidal groups, while Hewitt–Ross strengthens the construction by introducing additional separation techniques to ensure that the original space remains closed in the free group. Elfard's construction follows the same product embedding philosophy but requires substantial modifications because the inversion operation is not assumed to be continuous. In particular, the separation of reduced words and the verification of the universal property must be achieved using suitable paratopological groups rather than the topological group tools available in the classical setting.

Thus, product embeddings provide a common framework connecting the three constructions. The differences between them arise not from the general embedding strategy, but from the algebraic-topological properties of the ambient category in which the free object is constructed.

8. Conclusion

Future research may investigate whether the product embedding methods of Kakutani, Hewitt–Ross, and Elfard can be adapted to the category of quasitopological groups. Although free quasitopological groups have recently been constructed by Brazas and Emery [16], no analogous product embedding construction is currently known. Developing such a construction, and extending these methods to free semitopological groups, would

provide a natural continuation of the classical theory of free topological and free paratopological groups.

References

- [1] A. A. Markov, "On free topological groups," *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya*, vol. 9, no. 1, pp. 3–64, 1945.
- [2] S. Kakutani, "Free topological groups and infinite direct product topological groups," *Proc. Imperial Academy (Tokyo)*, vol. 20, pp. 595–598, 1944.
- [3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, vol. I, *Die Grundlehren der mathematischen Wissenschaften*, vol. 115. Berlin, Germany: Springer-Verlag, 1963.
- [4] C. Romaguera, M. Sanchis, and M. Tkachenko, "Free paratopological groups," *Topology Proc.*, vol. 27, no. 2, pp. 613–640, 2003.
- [5] O. V. Ravsky, "Paratopological groups. I," *Mat. Stud.*, vol. 16, no. 1, pp. 37–48, 2001.
- [6] A. Ravsky, "On the metrizability of topological groups," *Visnyk Lviv Univ., Ser. Mech. Math.*, vol. 61, pp. 82–92, 2004.
- [7] N. Pyrch and A. Ravsky, "Free paratopological groups," *Topology Appl.*, vol. 194, pp. 108–118, 2015.
- [8] A. S. Elfard and P. Nickolas, "On the topology of free paratopological groups," *Bull. London Math. Soc.*, vol. 44, pp. 1103–1115, 2012.
- [9] A. S. Elfard and P. Nickolas, "On the topology of free paratopological groups. II," *Topology Appl.*, vol. 160, pp. 220–229, 2013.
- [10] A. S. Elfard, "Free paratopological groups," *Appl. Gen. Topol.*, vol. 16, no. 2, pp. 89–98, 2015, doi: 10.4995/agt.2015.1874.
- [11] A. S. Elfard, "Free paratopological groups. II," *Topology Proc.*, vol. 43, pp. 331–339, 2014.
- [12] A. S. Elfard, "Neighborhood bases at the identity of free paratopological groups," *Topol. Algebra Appl.*, vol. 1, pp. 31–36, 2013.
- [13] R. Ellis, "A note on the continuity of inversion in a group," *Proc. Amer. Math. Soc.*, vol. 8, pp. 372–373, 1957.
- [14] A. V. Arhangel'skii and M. G. Tkachenko, *Topological Groups and Related Structures*, *Atlantis Studies in Mathematics*, vol. 1. Paris, France: Atlantis Press, 2008.
- [15] J. L. Kelley, *General Topology*. Toronto, ON, Canada: D. Van Nostrand Company, Inc., 1955.
- [16] J. Brazas and S. Emery, "Free quasitopological groups," *Topology Appl.*, vol. 326, Art. no. 108416, 2023.